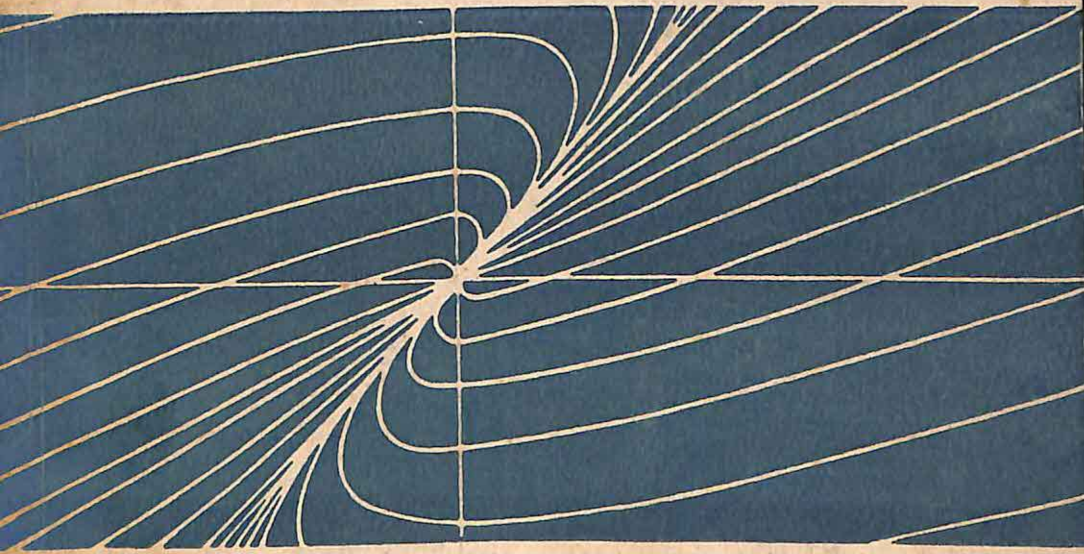
COACHING AT YOUR HOME
SERIES

PHYSICS

MECHANICS



A GOLDEN GUIDE TO COMPETITIVE EXAMINATIONS

for

IIT's, CEE (Bihar), ISM (Dhanbad),
PMDT, BIT (Mesra), IIT (Roorkee),
JEE (West Bengal), NDA etc.

BHARATI
BHAWAN
(P&D)

COACHING AT YOUR HOME

PHYSICS

(Mechanics)

COACHING AT YOUR HOME SERIES

(A Golden Guide to Competitive Examinations)

PHYSICS

(MECHANICS)

[for IITs, CEE (Bihar), ISM (Dhanbad), PMDT, BIT (Mesra),
IIT (Roorkee), JEE (W.B.) NDA etc.]

by

Prof. N. N. Ghosh
Reader, Dept. of Physics
St. Xavier's College, Ranchi.

&

Dr. D. Mukherjee
Reader, Dept. of Physics
B. N. College, Patna

Written in Consultation with Coaching
Experts of reputed Coaching Institutes

S. C. E. R. T. Library
Calcutta

BHARATI BHAWAN
PUBLISHERS & DISTRIBUTORS

Published by
BHARATI BHAWAN (Publishers & Distributors)
Govind Mitra Road, Patna-800 004

Branches :

RANCHI, 20, Jail Road (East), Tharpakhna.
DARBHANGA, New Market, Tower Chowk.
MUZAFFARPUR, Banka Bazar, Motijhil.
CALCUTTA, Room No. C-11 (First Floor), College Street Market.

Agencies :

PATNA, B. B. Agency, G. M. Road.
BHAGALPUR, Bharati Bhawan (Agency), Dr. Rajendra Prasad Road.
JAMSHEDPUR, Sen Book Stores, Godown No. 8, Tank Area, Sakchi.
DHANBAD, Bidya Bhawan, Rly. Cinema Road.
GAYA, Universal Book Agency, Choudhury Compound, G. B. Road.
CALCUTTA, Gyan Bharati Prakashan, 171/A, M. G. Road.
CALCUTTA, Indian Publishing House, 22/1, Bidhan Sarani.

© Authors

S.C.E.R.T. W.B. LIBRARY
Date 23.12.98
Accn. No.. 9782.

First edition, 1986

S.C.E.R.T. W.B. LIBRARY
Date 23.12.98
Accn. No.. 9782.

Rs. 32·75

Printed at Tapan Printing Press, Patna-4.

PREFACE

The series of books **Coaching at Your Home** (A Golden Guide to Competitive Examinations) are primarily aimed at fulfilling the long felt need of authentic publications, reasonably priced, for the students aspiring to compete at the competitive examination of Bihar, Roorkee, C.E.E. Bihar, I.I.T. and other important entrance examinations, for Engineering Colleges. Our further intention is to provide a suitable and effective alternative to the expensive and time bound coaching imparted at different Coaching Institues. The books in the series have been planned to cater to the student's requirement of sufficient self learning and practicing materials, with a personalised touch of a teacher for COACHING AT THEIR HOME.

The students, in general, aspiring to go in for an engineering entrance examination, find themselves at sea while tackling the syllabus or the questions of the above examinations, either because they have been neglecting their regular studies or because of the previous knowledge imparted to them at earlier levels were not adequate. But the entrance examinations require them to have a very systematic knowledge of the subject, so that they may be able to solve new problems similar to those they have already done. This presents the need of good, systematic and simplified guide book which can impart knowledge as well as guide the students in the right direction to tackle and solve new problems to achieve success.

The aim of writing this guide in physics is to provide coaching at home to those students who cannot take correspondence coaching or attend regular coaching institutes. Keeping these in mind, the present book on Physics (Mechanics) has been prepared.

The entire subject matter has been arranged in several lessons and each lesson contains resume of the subject matter, critical questions, numerical and subject matter objectives, examples and exercises. At the end of each lesson a set of test questions on the topics in the lesson is given to test one's depth of understanding. At the end of all the lessons, several sets of model questions of different competitive examinations such as IIT, CEE, BIT (Mesra), JEE (WB), Roorke, ISM (Dhanbad), NDA etc., are given with the same motto—test yourself.

To derive maximum benefit from this book one needs a basic book giving conceptual ideas. *Introductory Physics Part I & II* authored by N. N. Ghosh will serve the purpose. Proceed with chapters of the basic book and then take up the same chapters of this book. Though answers of all the problems are provided students are strongly advised not to refer to them hurriedly. They should consult the solutions only as a last resort. Before taking up the lessons carefully read GUIDELINES FOR SOLVING PROBLEMS and GUIDELINES FOR BEHAVIOUR IN THE EXAMINATION HALL appearing in following pages.

—Authors

GUIDELINES FOR SOLVING PROBLEMS

● 1. Always remember, that problems are based on basic and simple principles. The examiner may make problems more difficult or lengthy by bringing in diverse conditions. You have to think out the basic principles. So, first of all, read the problem carefully, think deeply and then proceed to solve the problem only after realising its underlying principles.

● 2. Wherever possible, specially in mechanics, light and current electricity, draw a diagram elucidating the essence of the problem; this will always help you to discover the fundamental principle involved.

● 3. Never consider any word redundant in any problem. For example, suppose a problem reads—'A uniform rod is tied by a massless string..........' It is a common error to think that rods are essentially uniform and strings massless and the words 'uniform' and 'massless' are redundant. This is not true. It is the word 'uniform' that permits us to take the cg of the rod at its centre and the word 'massless' makes the problem much simpler. So, note the significance of each word of the problem.

● 4. Laws, equations and formulae may have limitations. Try to learn their limitations. Before applying them be careful about the applicability to your specific problem. Suppose you are asked to find the velocity acquired by a body released from a height 10^4 km. If you apply the well-known formula $v^2-u^2=2gs$, the result will be totally wrong because the formula is applicable only when the acceleration is uniform. But here the acceleration due to gravity is not uniform over such a large distance above the surface of the earth. This fact restricts the application of the above formula to this particular problem.

● 5. If you are unable to solve a problem on the first attempt, don't hurry to consult the solution provided. Make a fresh study of the theoretical aspects, paying attention to finer points. Experience shows that a repeated study of the theory with definite a problem in mind is very effective and ensures quick success, when a second attempt at the problem is made. Failure in the first attempt should never cause gloom and despair. You have equal potentiality and creative abilities as the other successful students.

●6. In most cases the problem should be solved in a general form with suitable notations (symbols) for the relevant quantities. However, this should not be considered as a rule. Sometimes the price of generality is an excessive volume of calculations. The reverse is also true with direct numerical values. It becomes cumbersome in forwarding big numerical figures in successive steps. You have to apply your sense of discretion to decide what process would be convenient. *Do not forget to give proper unit in the final answer.*

●7. Do not be disturbed if you think that some essential quantities are not specified in the statement of the problem—perhaps they will cancel out. Universal constants will generally be supplied. Do not be disturbed if one or two constants are missing. The values of universal constants are supplied at the end of this book.

GUIDELINES FOR BEHAVIOUR IN THE EXAMINATION HALL

●1. Read the entire question paper carefully and mark the questions and problems that appear familiar to you or whose solutions are almost apparent to you. First try all such questions and problems.

●2. Remember that all competitive examinations are the processes of elimination and not of merit scholarship. So do not be discouraged by low score in the first attempt. Next take up the unfamiliar questions and problems.

●3. Solve each problem, as a rule, in the general form, that is in letter notations and finally substitute the numerical values. It becomes cumbersome to carry over numerical values in many successive steps.

●4. Do not indulge in lengthy arithmetical simplifications to obtain the most accurate numerical value up to several places of decimals. Make use of significant figures and rules of approximation to arrive at the final numerical value in the shortest possible time.

●5. If there is negative marking for wrong answers, it is advisable not to indulge in 'guess answers'.

CONTENTS

LESSON-1 1

Units and Dimensions
Vectors and Scalars
Kinematics (Analytical and Graphical)
Relative Velocity

LESSON-2 66

Laws of Motion
Free-Body Diagrams
Momentum : Principle of Conservation of Momentum
Motion of Connected Bodies
Friction : Limiting and Static

LESSON-3 141

Moment of a Force and Torque
Centre of Mass and Centre of Gravity
Machines
Equilibrium of Bodies

LESSON-4 184

Work, Power, Energy
Principle of Conservation of Energy : Work Energy Theorem
Collision of Bodies : Elastic and Inelastic

LESSON-5 234

Circular Motion
Simple Harmonic Motion

LESSON-6 310

Gravitation
Simple Pendulum
Elasticity

LESSON-7 375

Rotational Dynamics
Moment of Inertia
Conservation of Angular Momentum

LESSON-8 414

Hydrostatics and Hydrodynamics

LESSON-9 450

Surface Tension
Errors and their Measurement

□□□

List of Important Formulae

1. Kinematics :

Average velocity and average acceleration

$$<\vec{v}> = \frac{\triangle \vec{r}}{\triangle t} \qquad <\vec{a}> = \frac{\triangle \vec{v}}{\triangle t}.$$

Uniform velocity and acceleration

$$\vec{v} = \frac{d\vec{r}}{dt} \qquad \vec{a} = \frac{d\vec{v}}{dt}.$$

Distance covered

$$x = \int v_x dt = \text{area of } v_x - t \text{ curve and } t-\text{axis}.$$

Equations of kinematics

Linear kinematics	*Rotational kinematics*
$v = v_0 + \mathrm{at}^2$	$\omega = \omega_0 + \alpha t$
$x = v_0 t + \frac{1}{2}\,\mathrm{at}^2$	$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 - v_0^2 = 2ax$	$\omega^2 - \omega_0^2 = 2\alpha\theta.$

Relative velocity : $\vec{v}_{AB} = \vec{v}_A - \vec{v}_B.$

2. Fundamentals of Dynamics :

$$\vec{F} = m\,\vec{a} \text{ or } m\frac{d\vec{v}}{dt} \text{ or } \frac{d\vec{p}}{dt} \qquad \text{where} \quad \vec{p} = m\,\vec{v}.$$

For a rigid body or system of particles

$$\vec{F}_{ext} = M\,\vec{a}_{cm} \text{ or } M\frac{d\vec{v}_{cm}}{dt}$$

For a system of varying mass such as rocket

$$F_{ext} = m\frac{dv}{dt} + u_{rel}\frac{dm}{dt}.$$

Forward thrust generated by leaving mass $= u_{rel}\dfrac{dm}{dt}$.

Acceleration of two connected bodies by a single thread passing over a frictionless pulley

$$a = \frac{m_1 - m_2}{m_1 + m_2} \cdot g.$$

$$T\,(\text{tension}) = \frac{2m_1m_2}{m_1+m_2}.g.$$

Moment of a force : $\tau = rF\sin\theta$

Vectorially, $\vec{\tau} = \vec{r}x\vec{F}.$

Equilibrium of a body : $\Sigma\vec{F} = 0$ and $\Sigma\vec{\tau} = 0.$

Limiting frictional force : $f_{lim} = \mu_s N.$

Work done by a force : $W = \int F ds\cos\theta$ or $\int \vec{F}.\vec{ds}.$

Power of an agent : $P = \vec{F}_{agent}.\vec{v}.$

Kinetic energy : $K = \frac{1}{2}mv^2.$

Gravitational potential energy : $U = U_0 + mgh.$

Strain potential energy of a spring $= \frac{1}{2}kx^2.$

3. Time period of a simple pendulum :

$$T = 2\pi\sqrt{\frac{l}{g}}.$$

Time period of a spring controlled pendulum :

$$T = 2\pi\sqrt{\frac{m}{k}}.$$

4. Dynamics of circular motion :

Centripetal force $= \frac{mv^2}{r}$ or $m\omega^2 r.$

Inclination of a cyclist with the vertical :

$$\tan\theta = \frac{v^2}{rg}.$$

Gravitation : $F = G\frac{m_1m_2}{r^2}.$

5. Gravitational field and potential :

Point mass	$g = -G\frac{m}{r^2}.$	$V = -G\frac{m}{r}.$
Shell		
(i) inside	$g = 0$	$V = -G\frac{M}{a}.$

(ii) outside $g=-G\frac{M}{r^2}$. $V=-G\frac{M}{r}$.

Solid sphere

(i) inside $g=-\frac{GM}{a^3}r$. $V=-\frac{GM}{2a^3}(3a^2-r^2)$.

6. Rotational dynamics of a solid body :

Equation of motion : $\tau=Ix\alpha$.
Angular momentum : $L=I\omega$.
Rotational kinetic energy $K=\frac{1}{2}I\omega^2$.

Work done : $W=\int\tau d\theta$.

Moment of inertia of some bodies :
Thin rod .... $I=\frac{1}{12}Ml^2$
Disc .... $I_{cm}=\frac{1}{2}Mr^2$.
Shell .... $I_{cm}=\frac{2}{3}Mr^2$.
Sphere .... $I_{cm}=\frac{2}{5}Mr^2$.
Theorem of perpendicular axes : $I_z=I_x+I_y$.
Theorem of parallel axes : $I=I_{cm}+Mh^2$.

7. Simple Harmonic Motion :

Displacement $x=a\sin(\omega t+\alpha)$.

Velocity $v=a\omega\cos(\omega t+\alpha)=\omega\sqrt{a^2-x^2}$.
Acceleration $f=-a\omega^2\sin(\omega t+\alpha)=-\omega^2x$.
Kinetic energy $K=\frac{1}{2}mv^2=\frac{1}{2}m\omega^2(a^2-x^2)$.
Potential energy $U=\frac{1}{2}m\,a^2\omega^2x^2$.
Total energy $E=\frac{1}{2}ma^2\omega^2$.

8. Elasticity :

Modulus of elasticity $=\frac{\text{stress}}{\text{strain}}$.

Strain potential energy density $=\frac{1}{2}\times$ stress $\times$ strain.
Elastic force in a spring $=Kx$
Strain potential energy of a spring $=\frac{1}{2}kx^2$.

9. Pressure at a point : $p=p_0+\rho gh$

Bernoulli's theorem : $\frac{1}{2}v^2+gh+\frac{p}{\rho}=$ a constant.

Equation of continuity : $\alpha_1v_1=\alpha_2v_2=\alpha_3v_3=\ldots\ldots\ldots$.

□□□

LESSON 1

Topics : **Units and Dimensions**
Vectors and Scalars
Kinematics (Analytical and Graphical)
Relative Velocity

1.1 Units and Dimensions

Units. To measure a physical quantity and express its unit we require a standard of that physical quantity. This 'standard' is called 'unit' of that physical quantity. Measure of any physical quantity $= nu$ where $n =$ numerical value of the physical quantity, $u =$ unit of the quantity.

Q. 1. *Do you think a definition of a physical quantity for which no method of measurement is known or given has meaning* ?

The units of the basic physical quantities are called base units and the units based on base units are called derived units. In SI there are seven base physical quantities—length, mass, time, temperature, current, luminous intensity and the amount of substance. The base units are—

(i) the metre (m), the standard of length
(ii) the kilogramme (kg), the standard of mass
(iii) the second (s), the standard of time
(iv) the kelvin (K), the standard of temperature
(v) the ampere (A), the standard of current
(vi) the candela (cd), the standard of luminous intensity
(vii) the mole (mol), the standard of the amount of substance.

In addition to base units two more units are defined in SI. These are called supplementary units. The radian (rad) and the steradian (sr) are the two supplementary units which are units for plane angle and solid angles respectively. For definitions of base units and supplementary units see Introductory Physics-Part I of Bharati Bhawan, Patna-4, Bihar.

Some of the Most Common Derived Units in SI with Symbols

Force	newton	N
Work or energy	joule	J
Power	watt	W

Torque	newton metre	Nm
Pressure	pascal	Pa
Frequency	hertz	Hz
Luminance	lux	lx
Charge	coulomb	C
Potential	volt	V
Capacitance	farad	F
Magnetic flux	weber	wb
Magnetic induction	tesla	T
Inductance	henry	H

Q. 2. *Is the choice of mass, length and time as fundamental quantities unique ?*

SI Prefixes and Multiplication Factors

Factor	*Prefix*	*Symbol*
10^{12}	tera	T
10^{9}	giga (billion)	B
10^{6}	mega (million)	M
10^{3}	kilo	k
10^{-3}	milli	m
10^{-6}	micro	μ
10^{-9}	nano	n
10^{-12}	pico	p
10^{-15}	femto	f
10^{-18}	atto	a

Some Important Auxiliary Units

Angstrom (A) $= 10^{-10}$ m
Are (a) $= 100$ m^2
Fermi (fm) $= 10^{-15}$ m
Barn (b) $= 10^{-28}$ m^2
Astronomical Unit (A. U.) $= 1{\cdot}496 \times 10^{11}$ m
Parsec $= 3{\cdot}048 \times 10^{16}$ m
Lightyear $= 9{\cdot}460 \times 10^{15}$ m.

Q. 3. *When human race would settle on other planets, what drawbacks would our standards of length and time have ? What drawbacks would atomic standards have ?*

Q. 4. *Can you suggest a method to measure* (a) *radius of the earth,* (b) *distance between the sun and the earth.*

Dimensions. The dimensions of a physical quantity are the powers of the fundamental (base) quantities to which they are to be raised to represent a unit of that physical quantity. The dimension of the base quantities are denoted by L (length), M (mass), T (time), K (temperature), I (current), P (luminous intensity) and mol (mole). The symbol $[A]$ is read as 'dimension of A'. The dimensions of a physical quantity are calculated either from its definition or the formula in which it occurs.

Dimensions of Some Common Physical Quantities

Velocity	LT^{-1}
Acceleration	LT^{-2}
Force	MLT^{-2}.

Q. 5. *How many physical quantities can be connected by the method of dimensions in mechanics ?*

1.2 Scalars and Vectors

Scalars. A scalar quantity is that which requires only magnitude for its complete specification, e.g., mass, time, volume, temperature, speed, energy, current, electric and magnetic flux etc.

Vectors. Vectors are those which require both magnitude and direction for their complete specification, e g., displacement, velocity, acceleration, angular velocity, magnetic field, current density, etc. In addition to the above two specifications viz. magnitude and direction, a vector must obey the rules of vector algebra given below.

Frame of Reference. To locate the position of a particle or to represent a vector geometrically, we need a fixed point called the 'origin' $[O]$, and a set of three mutually perpendicular lines OX, OY, OZ meeting at the origin. This is called the frame of reference. In a frame of reference a vector is represented by an arrowed straight line (e.g. AB) whose length is proportional to the magnitude of the vector and the arrow on it beams in the direction of the vector.

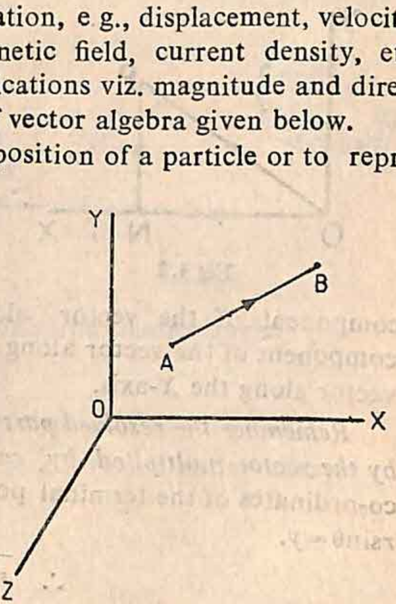


Fig. 1.1

The frame of reference is usually chosen as shown in Fig. 1.1. However, in many simple cases, we need to consider only two axes. Usually, OX and OY are used in such cases, e.g., Fig. 1.2.

Q. 6. *What does an observer on the earth mean by 'up' and 'down'? Does all such observers use the same frame of reference?*

Vector Addition and Subtraction. In any triangle ABC

$$\vec{AB}+\vec{BC}=\vec{AC}.$$

This is called 'triangle law' of vector addition. Vectors are added by this law. Subtraction of a vector $\vec{b}$ from $\vec{a}$ is addition of vector $-\vec{b}$ to the vector $\vec{a}$.

Representation of a Vector in terms of Unit Vector.

$$\vec{a} = a \,.\, \hat{a}$$

that is, a vector is always equal to the unit vector along it multiplied by its modulus.

Resolved Parts of a Vector. If a vector $\vec{r}$ is represented by $\vec{OP}$ in a two-dimensional frame of reference, then $\vec{OP}=\vec{r}$.

(Let $\hat{i}, \hat{j}, \hat{k}$ represent unit vectors in the directions OX, OY, OZ.)

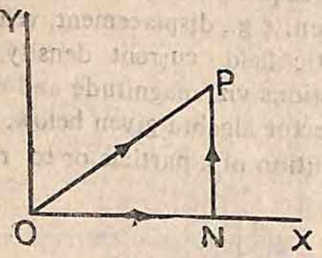


Fig 1.2

Draw perpendicular PN from P on OX. By the triangle law of vector addition we have

$$\vec{OP}=\vec{ON}+\vec{NP}.$$

If $\angle PON=\theta$, $ON=OP\cos\theta$, or

$\vec{ON}=OP\cos\theta\,\hat{i}$ and $\vec{NP}=OP\sin\theta\,\hat{j}$.

$$\therefore \vec{r}=r\cos\theta\,\hat{i}+r\sin\theta\,\hat{j}.$$

Here $r\cos\theta\,\hat{i}$ is called the vector component of the vector along X-axis and $r\cos\theta$ is the scalar component of the vector along X-axis or the resolved part of the vector along the X-axis.

Remember the resolved part of a vector along any direction is given by the vector multiplied by cosine of the angle. If x, y are the co-ordinates of the terminal point P of the vector then $r\cos\theta=x$ and $r\sin\theta=y$.

$$\therefore \vec{r}=x\hat{i}+y\hat{j}$$

and $r=\sqrt{x^2+y^2}$ and $\tan\theta=y/x$ or $\theta=\tan^{-1}y/x$.

In three-dimensional co-ordinate axes

$$\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$$

and $r=\sqrt{x^2+y^2+z^2}$

and $\tan\theta_x=\dfrac{\sqrt{y^2+z^2}}{x}$, $\tan\theta_y=\dfrac{\sqrt{z^2+x^2}}{y}$, $\tan\theta_z=\dfrac{\sqrt{x^2+y^2}}{z}$.

$\cos\theta_x$, $\cos\theta_y$ and $\cos\theta_z$ are called 'direction cosines' of the vector.

Scalar or Dot Product. The scalar product of two vectors is defined as $\vec{a}\,.\,\vec{b} = |\vec{a}|\;|\vec{b}|\cos\theta$ where θ is the angle (smaller one) between $\vec{a}$ and $\vec{b}$. The scalar product of two vectors is a scalar.

Characteristics of Scalar Products.

(i) $\vec{a}\,.\,\vec{b} = \vec{b}\,.\,\vec{a}$ (Commutative law)

(ii) $\vec{a}\,.\,(\vec{b}+\vec{c}) = \vec{a}\,.\,\vec{b}+\vec{a}\,.\,\vec{c}$ (Distributive law)

(iii) $\vec{a}\,.\,\vec{a} = a^2$

(iv) $\vec{a}\,.\,\vec{b} = a_xb_x+a_yb_y+a_zb_z$ where a_x, a_y, a_z and b_x, b_y and b_z are scalar components of $\vec{a}$ and $\vec{b}$ respectively.

(v) $\hat{i}\,.\,\hat{i}=\hat{j}\,.\,\hat{j}=\hat{k}\,.\,\hat{k}=1$ and $\hat{i}\,.\,\hat{j}=\hat{j}\,.\,\hat{k}=\hat{k}\,.\,\hat{i}=0$.

Vector Product. The vector product of two vectors $\vec{a}$ and $\vec{b}$ is defined as a vector $\vec{c}$ whose mod is given by $\vec{c} = |\vec{a}|\;|\vec{b}|\sin\theta$ and whose direction is given by the direction of translation of a screw perpendicular to $\vec{a}$ and $\vec{b}$ and rotating from $\vec{a}$ to $\vec{b}$ through the smaller angle.

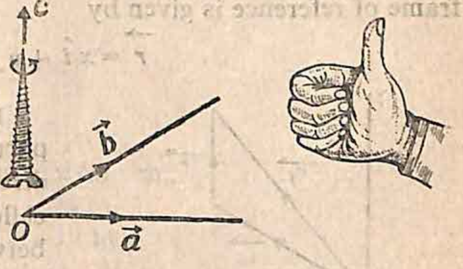


Fig. 1.3

Characteristics of Vector Products.

(i) $\vec{a}\times\vec{b} = -\,\vec{b}\times\vec{a}$ (Commutative law does not hold)

(ii) $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b}+\vec{a}\times\vec{c}$ (Distributive law)

(iii) $\vec{a}\times\vec{a} = 0$

(iv) $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$

(v) $\hat{i}\times\hat{j}=\hat{k}$, $\hat{j}\times\hat{k}=\hat{i}$ and $\hat{k}\times\hat{i}=\hat{j}$.

Types of Vectors.

(i) UNIT VECTOR—A vector whose mod (magnitude, modulus) is unity. It is represented by a hat (e.g. $\hat{i}$) instead of an arrow over it.

(ii) NULL VECTOR—A vector whose mod is zero.

(iii) CONSTANT VECTOR—A vector whose components are constants.

(iv) AXIAL VECTOR—Vectors associated with rotational motion are called axial vectors, e.g., angular velocity, torque. The direction of an axial vector is always along the normal to the plane of rotation and particularly along the direction of translation of a screw perpendicular to the plane of rotation and rotating in the same direction.

Q. 7. *Is division of vectors defined ?*

1.3 Kinematics

Position Vector. The distance of a particle specified in magnitude and direction from the origin of a frame of reference is called the 'position vector' of the particle. The position vector of a point in a frame of reference is given by

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}.$$

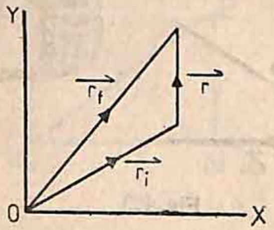


Fig. 1.4

Displacement. The distance of a particle specified in magnitude and direction from the initial position is called displacement of the particle between the intial and final position.

$$\vec{r} = \vec{r}_f - \vec{r}_i.$$

Uniform Velocity and Acceleration

Uniform velocity means motion in a straight line with constant speed (zero acceleration). Uniform acceleration means constant acceleration in a fixed direction.

$$\vec{v} = \frac{d\vec{r}}{dt} \quad \text{and} \quad \vec{a} = \frac{d\vec{v}}{dt}.$$

When motion takes place along a straight line, in the X direction, $v = \frac{dx}{dt} = (\dot{x}) =$ slope of x-t plot. Here x is the displacement of the particle along the line from a fixed point on the line.

For brevity $\frac{dx}{dt}$ is sometimes written as $\dot{x}$ and $\frac{d^2x}{dt^2}$ as $\ddot{x}$ etc.

$$a = \frac{dv}{dt} = (\ddot{x}) = \text{slope of } v\text{-}t \text{ plot.}$$

Distance Covered.

$s = \int v dt =$ area between v-t curve and t-axis.

Note. Distance covered here means displacement. The distance described is the sum of distances described irrespective of their signs (positive or negative).

Q. *The velocity of a particle is plotted against time, as shown. Find* (a) *the distance described by the particle and* (b) *the displacement of the particle.*

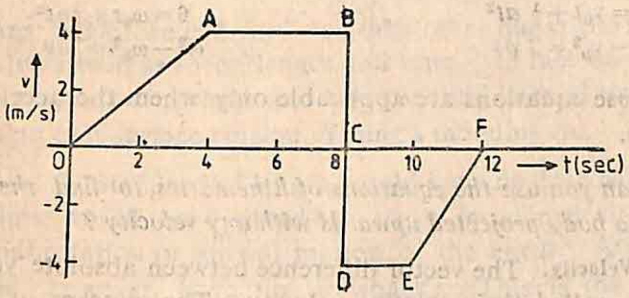


Fig. 1.5

Ans. The unit of area here is (m/s) × (s) = m.
The positive area of $OABC = 24$ m.
The negative area of $CDEF = 12$ m.

(a) The total area (neglecting sign) = 36 m.
This is the distance described.

(b) The total area (considering sign) = 12 m.
This is the displacement of the particle.

Q. *In the previous question, find the acceleration for OA and EF. Suggest what happens at BD.*

Ans. Since OA is a straight line, it suggests constant acceleration (slope is constant). The acceleration is given as

$$a = \text{slope of } OA = \frac{4 \text{ m/s}}{4 \text{ s}} = 1 \text{ m/s}^2.$$

Similarly, the acceleration at EF is also uniform, equal to

$$a = \text{slope of } EF = \frac{4 \text{ m/s}}{2 \text{ s}} = 2 \text{ m/s}^2.$$

At BD, the velocity changes from 4 m/s to -4 m/s instantaneously, that is, it reverses in direction without change in magnitude.

This can occur in an elastic collision with a rigid body.

Angular Velocity and Acceleration of a Solid Body.

$$\omega = \frac{d\theta}{dt}, \quad \alpha = \frac{d\omega}{dt}.$$

Relation of Angular Velocity and Speed.

$$v = \omega r.$$

Equations of Kinematics.

Equations of linear kinematics	*Equations of rotational kinematics*
$v = v_0 t + at$	$\omega = \omega_0 + \alpha t$
$s = v_0 t + \frac{1}{2} at^2$	$\theta = \omega_0 t + \frac{1}{2} \alpha t^2$
$v^2 - v_0^2 = 2as$	$\omega^2 - \omega_0^2 = 2\alpha\theta.$

Note. These equations are applicable only when the accelerations are uniform.

Q. 8. *Can you use the equations of kinematics to find the height attained by a body projected upwards with any velocity ?*

Relative Velocity. The vector difference between absolute velocities of bodies is called their relative velocity. The concept of relative velocity enables us to treat one body to be at rest while the other is in motion with the relative velocity. This formalism greatly simplifies many problems. If $\vec{V}_A$ is the absolute velocity a body A and $\vec{V}_B$ that of another body B, then relative velocity of A relative to B is

$$\vec{V}_{AB} = \vec{V}_A - \vec{V}_B$$

and

$$\vec{V}_{BA} = \vec{V}_B - \vec{V}_A.$$

To calculate relative velocity, follow this simple rule :—If two bodies A and B are in motion, and you wish to find the velocity of A relative to B, then superimpose (add) a velocity equal and opposite to the velocity of B, to both A and B. Now B will come to rest, while A will have two velocities : (i) its original velocity and (ii) the extra velocity you have supplied. Find their resultant. This gives the relative velocity of A with respect to B.

The principle we follow here is that "the relative velocity between two bodies remains unchanged by giving the same additional velocity to both bodies."

Q. 9. *Can relative velocity be greater than the absolute velocity ?*

Q. 10. *A boy sitting in car moving with constant velocity throws a ball straight up into the air. Will the ball fall behind him ? In front of him ? Into his hand ? What happens if the car accelerates forward or goes round a curve while the ball is in the air ?*

Q. 11. *A ball is thrown up. If the air resistance is taken into account and is supposed to be constant, will time of ascent be longer or shorter than the time of descent ?*

ANSWERS TO QUESTIONS IN THE TEXT

Ans. 1. No, because a physical quantity is defined as that one which is measurable.

Ans. 2. No, we can very well take other quantities as fundamental quantities such as force, length and time. In fact we take very often these three (force, length and time) as fundamental quantities e.g., in defining unit surface tension, Young's modulus etc.

Ans. 3. Standard of length would have no drawback but standard of time would have drawback as it is defined either in terms of diurnal rotation or annual motion of the earth. Atomic standards would, however, have no drawbacks. This is the reason why all standards have been defined in terms of characteristics of atoms.

Ans. 4. By 'triangulation method'. For details see Introductory Physics-Part I, chapter 1, 2nd edition.

Ans. 5. By 'up' an observer on the earth means 'away from the centre of the earth' and by 'down' he means 'towards the centre of the earth'. No, e.g., if two observers on the north pole and south are asked to take 'upward' as y-axis then their y-axes are just opposite.

Ans. 6. Four, by equating the powers of M, L and T.

Ans. 7. No.

Ans. 8. No, because the equations are applicable so long the acceleration is uniform. The acceleration due to gravity is uniform only near the surface of the earth.

Ans. 9. Yes, e.g., when two bodies move in opposite directions, the relative velocity of each is greater than individual velocity.

Ans. 10. Falls, into his hand because of 'physical independence of vectors' i.e. the forward velocity of the ball in air is the same as that of the car. But if the car is accelerated, the ball will fall behind him. When the car goes round a curve it will fall in front of him because the ball in air will move along the tangent to the curved path through a certain distance whereas the car will move along the curved path of the same length.

Ans. 11. Let F be the constant force on the body. Then for its upward journey we have, $mg+f=ma$ where a is the **retardation.**

or
$$a=g+\frac{F}{m}=g+g' \text{ where } g'=\frac{F}{m}.$$

Then $v_0=(g+g')t_{ascent}$ and $v_0^2=2(g+g')\,h$ where v_0 is the velocity of projection.

$$t_{ascent}=\sqrt{\frac{2h}{g+g'}}.$$

During downward journey, $mg-F=ma'$ where a' is acceleration of downward motion.

or
$$a'=g-\frac{F}{m}=g-g'$$

$$\therefore \quad h=0+\tfrac{1}{2}(g-g')t^2_{descent} \text{ or } t_{descent}=\sqrt{\frac{2h}{g-g'}}.$$

Obviously
$$t_{descent}>t_{ascent}.$$

OBJECTIVE QUESTIONS

1. If the acceleration due to gravity is 10 $\mathrm{ms^{-2}}$ and the units of length and time are changed to kilometre and hour respectively, then the numerical value of the acceleration is (a) 360000, (b) 72000, (c) 36000, (d) 129600.

Ans. $10\ \mathrm{ms^{-2}}=n\ \mathrm{km}\ h^{-2}$ or $n=129600$. So the answer is (d).

2. If a particle starting from rest describes 7 m in the 4th second, the distance described in the 5th second is
(a) 12 m, (b) 9 m, (c) 11 m, (d) 14 m.

Ans. We have $S_{n\text{th}}=S_n-S_{n-1}=V_0+\frac{1}{2}\,a\,(2n-1)$.

$$\therefore \quad 7=\tfrac{1}{2}a\times 7$$
$$S_{5\text{th}}=\tfrac{1}{2}a\times 9.$$

Hence $S_{5\text{th}}=9$. So the answer is (b).

3. If $\vec{a}=2\hat{i}-3\hat{j}+\hat{k}$, $\vec{b}=\hat{i}+\hat{j}-2\hat{k}$, the scalar product of the vectors is (a) -5, (b) $+5$, (c) -3, (d) $+3$.

Ans. $\vec{a}\cdot\vec{b}=(2\hat{i}-3\hat{j}+\hat{k}).(\hat{i}+\hat{j}-2\hat{k})$
$=2\times1+(-3)\times1+1\times(-2)=-3.$
So the answer is (c).

4. If the acceleration due to gravity is represented by unity in a system of units, and one second be the unit of time, the unit of length is (a) 9·8 m, (b) 1 m, (c) 98 m, (d) ·98 m.

Ans. $1\ \text{L}\text{s}^{-2}=9{\cdot}8\ \text{ms}^{-2}$ or $L=9{\cdot}8$ m. So the answer is (a).

5. If the force of attraction between two unit masses (1 kg each) 1 km apart is taken as unit force and the unit of energy in this system is called 'eluoj' (joule written in the reverse order), then the relation of 'joule' and 'eluoj' is (a) $1\ \text{joule}=6{\cdot}67\times10^{-17}$ eluoj, (b) $1\ \text{eluoj}=6{\cdot}67\times10^{-14}$ joule, (c) $1\ \text{eluoj}=10^{-4}$ joule, (d) 1 joule $=6{\cdot}67\times10^{-9}$ eluoj ($G=6{\cdot}67\times10^{-11}\ \text{Nm}^2\text{kg}^{-2}$).

Ans. Let us call the force in the new system 'notwen' (newton written in reverse order).

Then $1\ \text{notwen}=G\dfrac{1\ \text{kg}\times1\ \text{kg}}{(1\ \text{km})^2}$ or $G=10^6\ \text{notwen m}^2\text{kg}^{-2}$.

But $G=6{\cdot}67\times10^{-11}\ \text{N m}^2\ \text{kg}^{-2}$.

$\therefore\ 1\ \text{notwen}=6{\cdot}67\times10^{-17}$ newton.

$1\ \text{eluoj}=1\ \text{notwen}\times1\ \text{km}=6{\cdot}67\times10^{-14}\ \text{newton}\times\text{m}=6{\cdot}67\times10^{-14}$ joule.

So the answer is (b).

6. If the dot product of two vectors is the same as the mod of their cross product, the angle between the vectors is (a) 90°, (b) 45°, (c) 30°, (d) 0°.

Ans. We have $ab\cos\theta=ab\sin\theta$ or $\theta=45°$. So the answer is (b).

7. If $\vec{a}=4\hat{i}-3\hat{j}$ and $\vec{b}=6\hat{i}+8\hat{j}$, $|\vec{a}+\vec{b}|$ is (a) 5, (b) $5\sqrt{5}$, (c) 35, (d) $\sqrt{5}$.

Ans. $\vec{a}+\vec{b}=4\hat{i}-3\hat{j}+6\hat{i}+8\hat{j}=10\hat{i}+5\hat{j}.$

$$|\vec{a}+\vec{b}|=\sqrt{10^2+5^2}=5\sqrt{5}.$$

So the answer is (b).

8. A point traversed half a circle of radius R during time interval τ. Its mean speed is (a) $\dfrac{2R}{\tau}$, (b) $\dfrac{2\pi R}{\tau}$, c) $\dfrac{\pi R}{2\tau}$, (d) $\dfrac{\pi R}{\tau}$.

Ans. $<v>$, mean speed $=\dfrac{\text{total distance}}{\text{total time}}=\dfrac{\pi R}{\tau}$. So the answer is (d).

9. In the above example the modulus of the mean velocity is

(a) $\dfrac{2R}{\tau}$, (b) $\dfrac{2\pi R}{\tau}$, (c) $\dfrac{\pi R}{2\tau}$, (d) $\dfrac{\pi R}{\tau}$.

Ans. $<\vec{v}>$, average velocity $=\dfrac{R\hat{i}-(-R\hat{i})}{\tau}=\dfrac{2R}{\tau}\ \hat{i}.$

So the answer is (a).

10. A man runs to a goal post 80 m away and returns to the starting point in 10 minutes. (a) His displacement, and the distance covered by him are both zero, (b) his displacement is zero and the distance covered by him is 160 m, (c) his displacement is 160 m and the distance covered by him is also 160 m, (d) his displacement is zero and the distance covered by him is also zero.

Ans. (b).

11. A train *A* moves to the east with velocity 40 km per hour and a second train *B* moves to the north with velocity 30 km per hour. The relative velocity of *A* relative to *B* is (a) 70 kmh to the east, (b) 10 to the north-east, (c) 50 kph at some angle south of east, (d) 50 kph at some angle north of east.

Ans. Answer is (c).

12. If a vector is given by $\vec{a}=3\hat{i}+4\hat{j}$, the modulus of the vector is (a) 7, (b) $\sqrt{7}$, (c) 1, (d) 5.

Ans. Answer is (d).

13. Which one of the following represents uniformly accelerated motion (a) $x=\sqrt{\dfrac{t-a}{b}}$, (b) $x=\dfrac{t-a}{b}$, (c) $t=\sqrt{\dfrac{x-a}{b}}$, (d) $x\sqrt{t+a}$, where a and b are constants and x is the distance described.

Ans. A quadratic in t as is evident from $s=ut+\frac{1}{2}at^2$ reprerents uniformly accelerated motion. Hence the answer is (c).

14. In the arrangement shown in the figure, the ends *P* and *Q* of an unstretchable string move downwards with uniform speed *u*. Pulleys *A* and *B* are fixed. Mass *M* moves upwards with a speed (a) $2u\cos\theta$, (b) $u/\cos\theta$, (c) $2u/\cos\theta$, (d) $u\cos\theta$. **(I. I. T. 1982)**

Ans. This is a problem on constraints of motion. Here length of the string and the distance between the pulleys are fixed. Let $2a$ be the distance between the pulleys and let x and y be the instantaneous vertical downward distances of M and P (or Q) from the horizontal.

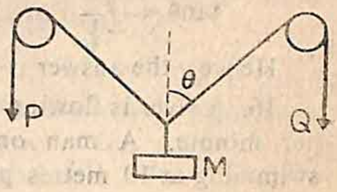


Fig. 1.6

Now from the constraint of the motion $2y+2\sqrt{a^2+x^2}=\text{a}$ constant, (because the string is unstretchable). Differentiating with respect to t we have

$$2\frac{dy}{dt}+2\frac{1}{2}\left(a^2+x^2\right)^{-2}.2x\frac{dx}{dt}=0.$$

Here $\quad \frac{dy}{dt}=u$ (given) and $\frac{dx}{dt}=v$ (say).

$$\therefore \quad u+\frac{x}{\sqrt{a^2+x^2}}\cdot v=0 \quad \text{or} \quad v=-\frac{u}{\frac{x}{\sqrt{a^2+x^2}}}=-\frac{u}{\cos\theta}.$$

The negative sign shows that the velocity of M is opposite to the velocity of P (or Q).

$\therefore \quad v=\frac{u}{\cos\theta}$. Hence the answer is (b).

15. A particle is moving eastward with a velocity of 5 ms^{-1}. In 10 s the velocity changes to 5 ms^{-1} northward. The average acceleration in this time is (a) zero, (b) $\frac{1}{\sqrt{2}}$ ms^{-2} towards north-west, (c) $\frac{1}{\sqrt{2}}$ ms^{-2} towards north-east, (d) $\frac{1}{2}$ ms^{-2} towards north-west.

(I.I.T 1983)

Ans. With reference to a frame x-axis eastward and y-axis northward, $\vec{v_i}=5\,\hat{i}$ and $\vec{v_f}=5\,\hat{j}$.

$$<\vec{a}>=\frac{\vec{v_f}-\vec{v_i}}{\triangle t}=\frac{5\,\hat{i}-5\,\hat{i}}{10}=-\tfrac{1}{2}\,\hat{i}+\tfrac{1}{2}\,\hat{j}.$$

$$\therefore \quad |<a>|=\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}=\frac{1}{\sqrt{2}}.$$

$\tan\theta = \frac{\frac{1}{2}}{-\frac{1}{2}}$ or $\theta = 135°$, that is, north-west direction.

Hence the answer is (b).

16. A river is flowing from west to east at a speed of 5 metres per minute. A man on the south bank of the river, capable of swimming at 10 metres per minute in still water, wants to swim across the river in the shortest time. He should swim in a direction (a) due north, (b) 30° east of north, (c) 30° west of north, (d) 60° east of north. **(I.I.T 1983)**

Ans. He should swim due north, because when he swims due north whole of his velocity is effective in taking him across the river. So the answer is (a).

17. In the above example if he wants to cross the river without being drifted he should swim (a) due north, (b) 30° east of north, (c) 30° west of north, (d) 60° east of north.

Ans. He should swim in such a direction that his resultant velocity is at right angles to the river. Answer is (c).

18. The dimensions of torque is (a) $M^2L^2T^{-2}$, (b) ML^2T^{-2}, (c) MLT^{-2}, (d) ML^2T^{-1}.

Ans. Torque = force × distance $= MLT^{-2} \times L = ML^2T^{-2}$. So the answer is (b).

19. The dimension of the gravitational constant is (a) ML^2T^{-3}, (b) $M^{-1}L^{-1}T^{-3}$, (c) ML^2T^{-2}, (d) $M^{-1}L^3T^{-2}$.

Ans. $[G] = \frac{[F][d^2]}{[m_1 \times m_2]} = \frac{MLT^{-2}L^2}{M^2} = M^{-1}L^3T^{-2}$.

So the answer is (d).

20. Parsec is the unit of (a) time, (b) angle, (c) distance, (d) velocity.

Ans. Answer is (c).

21. A femtometre stands for (a) 10^{-15} cm, (b) 10^{-6} m, (c) 10^{-5} m, (d) 10^{-15} m.

Ans. (d).

22. If $\vec{r} = a\hat{i} + b\hat{j}$, then the inclination of the vector with the x-axis in the anticlockwise direction is

(a) $\sin^{-1}\frac{b}{a}$, (b) $\sin^{-1}\frac{a}{b}$, (c) $\tan^{-1}\frac{b}{a}$, (d) $\tan^{-1}\frac{a}{b}$.

Ans. (c).

23. The slope of v-t curve gives (a) displacement, (b) total distance described, (c) acceleration, (d) average velocity.

Ans. Answer is (c).

24. In a rectilinear motion the distance described is given by (a) $\int v dt$, (b) slope $\left(\frac{dy}{dt}\right)$, (c) curvature $\left(\frac{d^2v}{dt^2}\right)$, (d) $\int t dv$.

Ans. Answer is (a).

25. If force (F), length (L), current (I) and time (T) are taken as bases then the dimensions of $\in_0$ is (a) $FL^2T^2T^{-2}$, (b) $F^{-1}L^2I^2T^2$, (c) $F^{-1}L^{-2}T^2I^2$, (d) $F^2L^2T^2I^2$.

Ans. We have, $$F=\frac{1}{4\pi\in_0}\frac{q_1q_2}{d^2}.$$

$$\therefore \quad [\in_0]=\frac{[q_1\times q_2]}{[F][d^2]}=\frac{IT\times IT}{FL^2}=F^{-1}L^{-2}T^2I^2.$$

So the answer is (c).

26. If a stone is released from a balloon rising with acceleration a at the instant when its velocity is v, then immediately after release the acceleration and velocity of the stone are (a) a(upward), v(upward), (b) g(upward), v(upward), (c) g(downward), v(upward), (d) $(g-a)$ and v(upward).

Ans. Answer is (c).

Note. A body A released from a body B will have the same initial velocity as B but *not* its acceleration. For example, a stone released from a balloon rising with acceleration a will have the same upward velocity as the velocity of the balloon when it is released but its acceleration will be g downward and not a.

27. The dimensions of capacitance are (a) $ML^{-2}T^2Q^{-2}$, (b) $M^{-1}L^2T^{-2}Q^2$, (c) $M^{-1}L^{-2}T^{-2}Q^2$, (d) $M^{-1}L^{-2}T^2Q^2$ where Q is the dimension of charge.

Ans. $$\text{Capacitance}=\frac{\text{charge}}{\text{potential difference}}.$$

$$\text{Potential difference}=\frac{\text{work done}}{\text{charge}}.$$

$$\therefore \quad \text{Capacitance}=\frac{\text{charge}^2}{\text{work done}}.$$

$\therefore$ [capacitance] $= \dfrac{Q^2}{ML^2T^{-2}} = M^{-1}L^{-2}T^2Q^2$.

So the answer is (d).

28. A vector of magnitude 10 units is in the south-east direction at inclination 37° with the x-axis, the vector representation of it with eastward as x-axis and northward as y-axis ($\sin 37° = {\cdot}6$, $\cos 37° = {\cdot}8$) is (a) $8\vec{i}+6\vec{j}$, (b) $6\vec{i}+8\vec{j}$, (c) $6\vec{i}-8\vec{j}$, (d) $8\,i-6\,j$.

Ans. (d).

29. Rain water is falling vertically downwards with a velocity v. When the velocity of the wind is zero water is collected at a rate R. When the wind starts blowing horizontally at a speed u, the rate of collection of water in the same vessel is

(a) $\sqrt{u^2+v^2}\,R$, (b) $\dfrac{v}{u}R$, (c) $\dfrac{uR}{v}$, (d) R.

Ans. In general R (rate of collection) $= vA\cos\theta$ where θ is the angle between the velocity of rain and the normal to the cross-section A of the vessel.

$$\therefore \quad R = v.A\cos 0° = A.v.$$

When the wind blows, $R' = v'.A\cos\theta$ where v' is the new velocity of rain. Now $v' = \sqrt{u^2+v^2}$ and $\cos\theta = \dfrac{v}{\sqrt{u^2+v^2}}$.

$$\therefore \quad R' = R.$$

So the answer is (d).

30. The range of a projectile is maximum when the angle of projection is (a) 30°, (b) 90°, (c) 45°, (d) 60°.

Ans. (c).

EXAMPLES

Units-dimensions and Vectors

1. Obtain the relation $t = 2\pi\sqrt{\dfrac{l}{g}}$ by the method of dimensions.

Sol. Let $t = kl^ag^b$. Taking dimensions of both sides we have $T = L^a(LT^{-2})^b$ or $T = L^{a+b}T^{-2b}$. $\therefore$ $-2b = 1$ and $a+b = 0$.

$$\therefore \quad a = \tfrac{1}{2} \text{ and } b = -\tfrac{1}{2}.$$

$\therefore$ $t = k\sqrt{\dfrac{l}{g}}$. Experimentally, $k = 2\pi$. $\therefore$ $t = 2\pi\sqrt{\dfrac{l}{g}}$.

2. Show that $p^{-\frac{5}{6}}\rho^{\frac{1}{2}}E^{\frac{1}{3}}$ where p is the pressure, ρ is the density and E is the energy of a bubble is of the dimension of time.

Sol. We have $[p^{-\frac{5}{6}}\rho^{\frac{1}{2}}E^{\frac{1}{3}}]=(ML^{-1}T^{-2})^{-\frac{5}{6}}(ML^{-3})^{\frac{1}{2}}(ML^{2}T^{-2})^{\frac{1}{3}}$

$$=M^{-5/6+1/2+1/3}L^{5/6-3/2+2/3}T^{+5/3-2/3}=M^{0}L^{0}T^{1}=T.$$

3. Find the dimensions of the electric permittivity.

Sol. The electric permittivity occurs in the formula

$$\triangle F=\frac{1}{4\pi\epsilon_0}\cdot\frac{q_1q_2}{r^2}.$$

$$\therefore\quad [\triangle F]=\frac{1}{[\epsilon_0]}\cdot\frac{[q]^2}{[r]^2}\quad\text{or}\quad MLT^{-2}=\frac{1}{[\epsilon_0]}\cdot\frac{(IT)^2}{L^2}$$

or $\quad [\epsilon_0]=M^{-1}L^{-3}T^{4}I^{2}$. *Ans.*

4. If a particle moves 30° north of east with velocity 10 m/s, represent its velocity vectorially in a frame of reference having eastward direction as x-axis and northward direction as y-axis.

Sol. x-component $=10\cos 30°=\dfrac{10\sqrt{3}}{2}=5\sqrt{3}$

and y-component $=10\cos 60°=5.$

$$\therefore\quad \vec{v}=5\sqrt{3}\,\hat{i}+5\,\hat{j}\quad \textit{Ans.}$$

5. If $\vec{a}=3\,\hat{i}-4\,\hat{j}$ and $\vec{b}=\hat{i}-3\,\hat{j}$, what is the angle between the two vectors ?

Sol. $\vec{a}\cdot\vec{b}=|\,\vec{a}\,|\;|\,\vec{b}\,|\cos\theta=5.\sqrt{10}\cos\theta.$

Also $\vec{a}\cdot\vec{b}=(3\,\hat{i}+4\,\hat{j}).(\,\hat{i}-3\,\hat{j}\,)=3.1+(-4)(-3)=15.$

$\therefore\quad 5\sqrt{10}\cos\theta=15\quad$ or $\quad\theta=\cos^{-1}3/\sqrt{10}.\quad$ *Ans.*

6. A particle suffers three displacements by 4 m in the northward, 2 m south-east and 1 m south-west. What is the displacement of the particle and what is the distance covered by it ?

Sol. Taking a frame of reference having x-axis in the eastward and y-axis in the northward, we have $\vec{s_1}=4\,\vec{j}$, $\vec{s_2}=2\cos 45°\,\vec{i}-2\sin 45°\,\vec{j}$ and $\vec{s_3}=-2\cos 45°\,\vec{i}-2\sin 45°\,\vec{j}.$

$$\vec{s}=\vec{s_1}+\vec{s_2}+\vec{s_3}=4\vec{j}+\sqrt{2}\,\vec{i}-\sqrt{2}\,\vec{j}-\sqrt{2}\,\vec{i}-\sqrt{2}\,\vec{j}$$

$$\therefore\quad \vec{s}=(4-2\sqrt{2})\,\vec{j}.$$

$|\vec{s}| = 4 - 2\sqrt{2} = 1{\cdot}17$ m (northward). *Ans.*

Total distance described $= 4 + 2 + 1 = 7$ m. *Ans.*

7. (a) The coordinates of the point P in a rectangular frame of reference are $(-2, 4)$ and that of another point Q are $(3, 5)$. Find the vector $\vec{PQ}$ and $\vec{QP}$.

Sol. Let O be the origin of the frame of reference. Then $\vec{OP} = -2\,\hat{i} + 4\,\hat{j}$ and $\vec{OQ} = 3\,\hat{i} + 5\,\hat{j}$.

$\vec{PQ} = \vec{PO} + \vec{OQ} = \vec{OQ} - \vec{OP} = (3\,\hat{i} + 5\,\hat{j}) - (-2\,\hat{i} + 4\,\hat{j}) = 5\,\hat{i} + \hat{j}$ *Ans.*

$\vec{QP} = -\vec{PQ} = -5\,\hat{i} - \hat{j}$. *Ans.*

7. (b) Show that $\vec{a}\cdot(\vec{b} + \vec{c}) + \vec{b}\cdot(\vec{c} + \vec{a}) + \vec{c}\cdot(\vec{a} + \vec{b}) = 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a})$.

Sol.
$$\begin{aligned}&\vec{a}\cdot(\vec{b} + \vec{c}) + \vec{b}\cdot(\vec{c} + \vec{a}) + \vec{c}\cdot(\vec{a} + \vec{b})\\ &= \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c} + \vec{b}\cdot\vec{a} + \vec{c}\cdot\vec{a} + \vec{c}\cdot\vec{b}\\ &= 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a})\end{aligned}$$

$(\because\ \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a},\ \vec{c}\cdot\vec{a} = \vec{a}\cdot\vec{c}$ and $\vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{b})$.

8. Find the cross product of the vectors $2\,\hat{i} + 3\,\hat{k}$ and $\hat{i} + \hat{j} - \hat{k}$.

Sol. Writing the first vector in the full form, viz., $2\,\hat{i} + 0\,\hat{j} + 3\,\hat{k}$, we have

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 3 \\ 1 & 1 & -1 \end{vmatrix} = -3\,\hat{i} + 5\,\hat{j} + 2\,\hat{k}\text{. } \textit{Ans.}$$

9. A train travels a distance 5 km in the north-east direction at a rate of 40 km/h and then a distance 5 km at the rate of 40 km/h in the south-east direction. What is the average speed during the whole journey and what is the average velocity ?

Sol. Total time of travel $= \frac{5}{40} + \frac{5}{40} = \frac{1}{4}$ h.

Total distance described $= 5$ km $+ 5$ km $= 10$ km.

$\therefore$ Average speed $= \frac{10}{\frac{1}{4}} = 40$ km/h. *Ans.*

Take a frame of reference with x-axis in the eastward and y-axis in the northward direction. $\vec{s}$, total displacement vector

$$= \vec{r_f} - \vec{r_i} = 5\sqrt{5}\,\hat{i} - 0\,\hat{i} = 5\sqrt{5}\,\hat{i}$$

$$<\vec{v}>, \text{ avearage velocity} = \frac{5\sqrt{5}\,i}{\frac{1}{4}} = 20\sqrt{5}\,\hat{i},$$

that is, average velocity is $20\sqrt{5}$ km/h to the east.

10. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are the position vectors of three collinear points A, B and C with respect to the origin O, show that $\vec{a} \times \hat{p} = \vec{b} \times \hat{p} = \vec{c} \times \hat{p}$ where $\hat{p}$ is the unit vector along the line ACB.

Sol. Draw perpendicular ON from O on ACB. Let $\hat{n}$ be the unit vector along the normal to the plane $OACBO$.

Then $\vec{a} \times \hat{p}$

$$= |\vec{a}|\;|\hat{p}|\;\sin OAN\,\hat{n}$$

$$= OA \sin OAN\,\hat{n}$$

$$= ON\,\hat{n}.$$

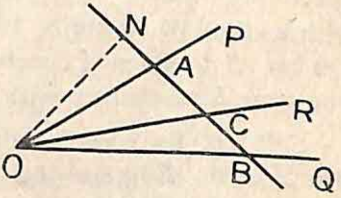


Fig. 1.7

Similarly, $\vec{b} \times \hat{p} = ON\,\hat{n}$ and $\vec{c} \times \hat{p} = ON\,\hat{n}$.

$\therefore\ \vec{a} \times \hat{p} = \vec{b} \times \hat{p} = \vec{c} \times \hat{p}$. *Proved.*

11. Forces P, Q act at a point O and have resultant R. If any transversal cuts their lines of action at A, B and C, show that

$$\frac{P}{OA} + \frac{Q}{OB} = \frac{R}{OC}.$$

Sol. Take O to be the origin. (refer to Fig. 1.7) Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be the position vectors of A, B and C respectively. Then the unit vectors along them are

$$\hat{a} = \frac{\vec{a}}{OA},\ \hat{b} = \frac{\vec{b}}{OB},\ \hat{c} = \frac{\vec{c}}{OC}$$

so that $$\vec{P} = P\,\hat{a} = \frac{P}{OA}\,\vec{a},\ \vec{Q} = \frac{Q}{OB}\,\vec{b},\ \vec{R} = \frac{R}{OC}\,\vec{c}.$$

Since $\vec{P}+\vec{Q}=\vec{R}$,

$$\therefore \quad \frac{P}{OA}\vec{a}+\frac{Q}{OB}\vec{b}=\frac{R}{OC}\vec{c}.$$

Taking cross-product with the unit vector $(\hat{p})$ along ABC we have

$$\frac{P}{OA}\vec{a}\times\hat{p}+\frac{Q}{OB}\vec{b}\times\hat{p}=\frac{R}{OC}\vec{c}\times\hat{p}.$$

Since A, B and C are collinear,

$$\vec{a}\times\hat{p}=\vec{b}\times\hat{p}=\vec{c}\times\hat{p} \qquad \text{(see the above example).}$$

$$\therefore \quad \frac{P}{OA}+\frac{Q}{OB}=\frac{R}{OC}. \quad \textit{Proved.}$$

Kinematics

1. A car A is travelling on a straight level road with a uniform speed of 60 km/h. It is followed by another car B which is moving with a speed of 70 km/h. When the distance between them is 2·5 km, the car B is given a deceleration of 20 km/h². After what distance and time B catches up with A ? (I.I.T. 1966)

Sol. Let us refer distances from the point where B is decelerated. Suppose the car B catches up with A in t hours. Then applying kinematic equations to the motion of A and B we have

$$x-2{\cdot}5=60t$$

and $$x=73t+\tfrac{1}{2}(-20)t^2.$$

Solving we have $t=\cdot5$ h and $x=32{\cdot}5$ km. *Ans.*

2. A body is projected vertically upwards from A, the top of a tower and reaches the ground in t_1 seconds. If it is projected vertically downwards from A with the same velocity, it reaches the ground in t_2 seconds. If it falls freely from rest at A, show that it would reach the ground in $\sqrt{t_1t_2}$ seconds.

Sol. Let us refer distances from the top of the tower, downward positive and upward negative.

Then $h=-v_0t_1+\frac{1}{2}gt_1^2$ from the second kinematic equation;
also $h=v_0t_2+\frac{1}{2}gt_2^2$
and $h=0.t+\frac{1}{2}gt^2$.

Eliminating h and v_0 from these equations we have $t=\sqrt{t_1t_2}$.

3. A car starts moving rectilinearly, first with acceleration a (the initial velocity is zero), then uniformly, and finally, decelerating at the same rate a, comes to rest. The total time of motion equals t seconds and the total distance covered equals s. How long does the car move uniformly ?

Sol. We have $s_1=\frac{1}{2}at_1^2$, $s_2=vt_2$ where v is the velocity acquired by the car after t_1 and t_2 is the time during which the car moves with uniforn velocity. We have to find this t_2.

Obviously $v=0+a\,t_1$.

$$\therefore \quad s_2=at_1t_2.$$

$$s_3=at_1.t_3-\tfrac{1}{2}at_3^2.$$

Since the car stops after t_3, $0=at_1-at_3$ or $t_1=t_3$.

$$\therefore \quad s_3=at_1^2-\tfrac{1}{2}at_1^2=\tfrac{1}{2}at_1^2.$$

It is given that $s=s_1+s_2+s_3=\frac{1}{2}at_1^2+at_1t_2+\frac{1}{2}at_1^2=at_1^2+at_1t_2$

and $\qquad t=t_1+t_2+t_3=2t_1+t_2 \quad (\because \quad t_1=t_3).$

Taking the value of t_1 in terms of t_2 from the last equation and putting in the equation above it we have

$$s=a\left(\frac{t-t_2}{2}\right)^2+a\left(\frac{t-t_2}{2}\right)t_2.$$

Hence $\qquad t_2=\left(t^2-\dfrac{4s}{a}\right)^{\frac{1}{2}}.$ *Ans.*

4. Two trains start simultaneously towards each other from two stations A and B which are 480 km apart. If the speeds of the two trains are 60 kph and 48 kph respectively, determine when the two trains meet after starting and at what distance from the station A.

Sol. Referring distances of both the trains from A we have

$$\frac{dx_1}{dt}=60 \text{ for the first train;}$$

$$\frac{dx_2}{dt}=-48 \text{ for the second train,}$$

it is negative because it is travelling in the negative direction.

Integrating, $\qquad x_1=60t+c,$

$$x_2=-48t+c'.$$

When $t=0$, $x_1=0$ but $x_2=480$;

$$\therefore \quad c=0 \text{ and } c'=480.$$

$$\therefore \quad x_1=60t \text{ and } x_2=480-48t.$$

When the trains meet, $x_1=x_2$.

$$\therefore \quad 60t=480-48t \quad \text{or} \quad t=4\tfrac{4}{9}\text{ h}=5\text{ h }27\text{ m.} \quad \textit{Ans.}$$

$$x_1=x_2=267\text{ km.} \quad \textit{Ans.}$$

S.C.E.R.T. W.B. LIBRARY
Date 23.12.98
Acc. No. 9782
S.C.E.R.T. Library Calcutta 25/3, B. C.

5. A balloon is ascending at a constant velocity of $10\ \text{ms}^{-1}$ when a body is released from it. The body reaches the ground in 8 secs. Find height of the balloon when the body is released. Also find the greatest height of the body above the ground.

Sol. Let us take a frame of reference at a height of the balloon when the body is released with vertical downward as y-axis and horizontal as$x$-axis. Applying the second kinematic equation (remember that *the body will have the same initial velocity as the balloon*) we have

$$h=(-10).8+\tfrac{1}{2}9{\cdot}8\times 8^2=253{\cdot}6\ \text{m.}\quad \textit{Ans.}$$

Let the body go up by h' above the x-axis of the frame

$$0-10^2=2x(9\ 8)\times(-h')\quad \text{or}\quad h'=5{\cdot}1\ \text{m.}$$

$\therefore$ The greatest height of the body

$$=h+h'=253{\cdot}6+5{\cdot}1=258{\cdot}7\ \text{m.}\quad \textit{Ans.}$$

6. The speed of a train increases at a constant rate α from zero to v and then remains constant for an interval and finally decreases to zero at a constant rate β. If l is the total distance decribed, prove that the total time taken is

$$t=\frac{l}{v}+\frac{v}{2}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right).$$

At what value of v, the time of travel is the shortest ? What is the value of the shortest time ?

Sol. This problem is best solved graphically, noting that distance described is equal to the area between the v-t curve and the t-axis. Let $OABC$ represent the v-t graph. Draw AD and BE perpendicular on the x-axis.

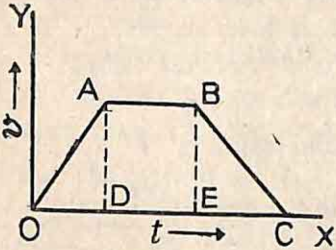


Fig. 1.8

$$\alpha=\text{acceleration}=\text{slope of } OA$$

$$=\frac{AD}{OD}=\frac{v}{OD}.$$

$$\therefore\quad OD=\frac{v}{\alpha}.$$

$$\beta=\text{retardation}=\text{slope of } BC=\frac{BE}{EC}=\frac{v}{EC}.$$

$$\therefore\quad EC=\frac{v}{\beta}.$$

l, distance described = area of the trapezium $OABC$

$$=\tfrac{1}{2}(AB+OC)AD$$

$$=\tfrac{1}{2}(AB+t)v \text{ where } t=OC\text{, total time of travel}$$

or $$AB=DE=\frac{2l}{v}-t.$$

$$\therefore\quad t=OD+DE+EC=\frac{v}{\alpha}+\frac{v}{\beta}+\frac{2l}{v}-t$$

or $$t=\frac{l}{v}+\frac{v}{2}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right).\ \textit{Proved.}$$

Now $$\frac{dt}{dv}=-\frac{l}{v^2}+\tfrac{1}{2}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right).$$

When t is minimum, $\frac{dt}{dv}=0$, $\therefore\ \frac{l}{v^2}=\frac{\alpha+\beta}{2\alpha\beta}$, or $v=\left(\frac{2\alpha\beta l}{\alpha+\beta}\right)^{\frac{1}{2}}$. *Ans.*

$$t_{min}=\frac{l}{\left(\frac{2\alpha\beta l}{\alpha+\beta}\right)^{\frac{1}{2}}}+\frac{\left(\frac{2\alpha\beta l}{\alpha+\beta}\right)^{\frac{1}{2}}}{2}\cdot\frac{\alpha+\beta}{\alpha\beta}=2\sqrt{\frac{(\alpha+\beta)l}{2\alpha\beta}}.\ \textit{Ans.}$$

7. Two particles start simultaneously from two points A and B with velocities 144 kph and 252 kph respectively in the same direction along the line AB and with accelerations 8 m/sec^2 and -4 m/sec^2 respectively. Show that the two particles meet twice. Find the timings of their meetings and the distances of their meeting points from A. $AB=36$ m.

Sol. 144 kph $=40$ ms^{-1} and 252 kph $=70$ ms^{-1}.

Let them meet at a distance x from A after time t.

Then $$x=40t+\tfrac{1}{2}8t^2$$
$$x-36=70t-\tfrac{1}{2}4t^2.$$

Subtracting, $t^2-5t+6=0$. Solving, $t=2$ s or 3 s. *Ans.*

Putting the values of t in either equation,

$$x=96 \text{ m and } 156 \text{ m. } \textit{Ans.}$$

8. Two ships are 10 km apart on the west-east line at some instant, the first one steaming to the east at 30 kph and the second one steaming to the north at 40 kph. What is the closest approach between them and after what time from that instant ?

Sol. With reference to a frame eastward x-axis and northward y-axis, $\vec{v_1} = 30\,\hat{i}$ and $\vec{v_2} = 40\,\hat{j}$.

$\vec{v}_{21}$ (relative velocity of 2 relative to 1)

$$= \vec{v_2} - \vec{v_1} = 40\,\hat{j} - 30\,\hat{i} = -30\,\hat{i} + 40\,\hat{j}.$$

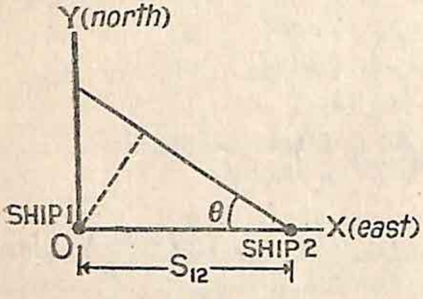


Fig. 1.9

From the concept of relative velocity 1 may be assumed to be fixed and 2 to be moving with velocity vector $(-30\,\hat{i} + 40\,\hat{j})$. This is a vector of magnitude 50 and inclined at $\theta = \tan^{-1} \frac{4}{3}$. The shortest distance is obviously the perpendicular distance of the relative velocity vector from the fixed position of ship 1.

$\therefore$ $S_{shortest} = S_{12}\sin\theta$ where S_{12} is the initial distance between 1 and 2

$$= 10 . \tfrac{4}{5} = 8 \text{ km. } \textit{Ans.}$$

t (time of cloest approach) $= \dfrac{S_{12}\cos\theta}{v_{21}} = \dfrac{10 \times \frac{3}{5}}{50} = \frac{3}{5}$ hour. *Ans.*

9. Two particles move in a uniform gravitational field with an acceleration g. At the initial moment the particles were located at some height above the ground at the same point and moved with velocities $v_1 = 3 \text{ ms}^{-1}$ and $v_2 = 4 \text{ ms}^{-1}$ horizontally in opposite directions. Find the distance between the particles at the moment when their velocity vectors become mutually perpendicular.

Sol. Let us take a frame of reference at the point of projection of the paticles with horizontal to the right as x-axis and vertical downward as y-axis. Let us designate the two particles as A and B. Then after time t,

$$\vec{v}_A = v_1\,\hat{i} + gt\,\hat{j} \quad \text{and} \quad v_B = -v_2\,\hat{i} + gt\,\hat{j}.$$

$\vec{v}_{AB}$ (relative velocity of A relative to B) $= \vec{v}_A - \vec{v}_B$

or $\vec{v}_{AB} = (v_1\,\hat{i} + gt\,\hat{j}) - (-v_2\,\hat{i} + gt\,\hat{j}) = (v_1 + v_2)\,\hat{i}$

$$\therefore \quad v_{AB} = v_1 + v_2.$$

Now $\vec{v}_A \cdot \vec{v}_B = -v_1v_2 + g^2t^2.$

When $\vec{v}_A$ and $\vec{v}_B$ are at right angles, $\vec{v}_A \cdot \vec{v}_B = 0.$

$$\therefore \quad -v_1v_2 + g^2t^2 = 0 \quad \text{or} \quad t = \frac{\sqrt{v_1v_2}}{g}.$$

Now, $S_{AB} = v_{AB} \times t$

$$\therefore \quad S_{AB} = (v_1 + v_2) \times \frac{\sqrt{v_1v_2}}{g} = \frac{(v_1+v_2)\sqrt{v_1v_2}}{g} = \frac{(3+4)\sqrt{3\times 4}}{9{\cdot}8}$$

$$= 2{\cdot}5 \text{ m. } \textit{Ans.}$$

10. A point moves along a straight line with deceleration which depends on the velocity as $\omega = a\sqrt{v}$, where a is a constant. At the initial moment the velocity of the point is equal to v_0. What distance does it traverse before it stops ? What time will it take to cover this distance ?

Sol. We have $\omega = -\dfrac{dv}{dt}$ (negative because ω is a deceleration).

$$\therefore \quad -\frac{dv}{dt} = a\sqrt{v} \quad \text{or} \quad \frac{dv}{\sqrt{v}} = -a\,dt.$$

Integrating, $2\sqrt{v} = -a\,t + a'.$

When $t=0$, $v=v_0$. $\therefore$ $2\sqrt{v_0} = a'.$

$$\therefore \quad 2\sqrt{v} = 2\sqrt{v_0} - at$$

or $\sqrt{v} = \sqrt{v_0} - \frac{1}{2}\,at$

or $v = v_0 - a\sqrt{v_0}\,t + \frac{1}{4}a^2t^2$

or $\dfrac{dx}{dt} = v_0 - a\sqrt{v_0}\,t + \frac{1}{4}a^2t^2.$

Integrating, $x = v_0t - \frac{1}{2}a\sqrt{v_0}\,t^2 + \frac{1}{12}a^2t^3 + x_0$ where x_0 is the constant of integration.

At $t=0$, $x=0$. $\therefore$ $x_0 = 0.$

$$\therefore \quad x = v_0t - \tfrac{1}{2}a\sqrt{v_0}\,t^2 + \tfrac{1}{12}a^2t^3.$$

When the point stops, $v=0$. $\therefore$ $0 = \sqrt{v_0} = \frac{1}{2}at$

or $t = \dfrac{2\sqrt{v_0}}{a}.$ ***Ans.***

$$\therefore \quad x=v_0\frac{2\sqrt{v_0}}{a}-\tfrac{1}{2}a\sqrt{v_0}\left(\frac{2\sqrt{v_0}}{a}\right)^2+\tfrac{1}{12}a^2\left(\frac{2\sqrt{v_0}}{a}\right)^3$$

$$=\frac{2}{a}v_0^{\frac{3}{2}}-\frac{2}{a}v_0^{\frac{3}{2}}+\frac{2}{3a}v_0^{\frac{3}{2}}=\frac{2v_0^{\frac{3}{2}}}{3a}. \quad \textit{Ans.}$$

11. A point travels along the x-axis with a velocity v_x which varies with time as shown in the figure. Assuming the coordinate of the point at $x=0$ at the moment $t=0$, draw the time dependence plots for the acceleration a_x, the x coordinate. What is its displacement at $t=7$ s ? What is the distance covered by it in 7 secs ?

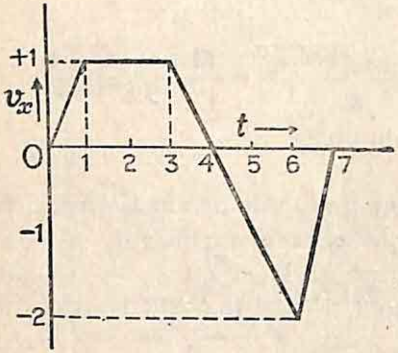


Fig. 1.10

Sol. In the interval 0-1 s,

$$a_x=\frac{dv_x}{dt}=\text{slope}=\frac{1}{1}=1.$$

In the interval 1-3 s,

$$a_x=\text{slope}=0.$$

In the interval 3-6 s,

$$a_x=\text{slope}=-1.$$

In the interval 6-7 s,

$$a_x=\text{slope}=2.$$

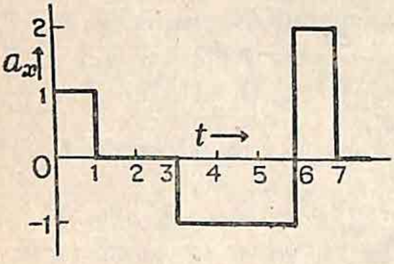


Fig. 1.11

The plot a_x against t in the whole range is shown in the Fig.

In the range **0-1 s,**

$$\frac{dv_x}{dt}=1.$$

$$\therefore \quad v_x=t+v_0.$$

At $t=0$, $v_x=0$. $\therefore \quad v_0=0$.

$$\therefore \quad v_x=t$$

or $\dfrac{dx}{dt}=t$ or $x=\dfrac{1}{2}t^2+x_0$.

At $t=0, x=0$. $\therefore \quad x_0=0$.

$$\therefore \quad x=\tfrac{1}{2}t^2. \qquad \text{.. (i)}$$

In the range 1-3 s, $\frac{dv_x}{dt}=0$ or $v_x=$ a constant $=1$. (from Fig. 1.10)

$$\therefore \quad \frac{dx}{dt}=1 \quad \text{or} \quad x=t+x_0.$$

At $t=1$, $x=\frac{1}{2}.1^2=\frac{1}{2}$ [substituting in (i)]

$$\therefore \quad \tfrac{1}{2}=1+x_0 \quad \text{or} \quad x_0=-\tfrac{1}{2}$$

$$\therefore \quad x=t-\tfrac{1}{2}. \qquad \text{.. (ii).}$$

In the range 3-6 s, $\frac{dv_x}{dt}=-1$

or $\quad v_x=-t+v_0$ (after integration).

At $t=3$, $v_x=1$. $\therefore$ $1=-3+v_0$ or $v_0=4$. $\therefore$ $v_x=4-t$

or $\quad \frac{dx}{dt}=4-t$

or $\quad x=4t-\frac{1}{2}t^2+x_0$ (after integration).

At $t=3$, $x=3-\frac{1}{2}=\frac{5}{2}$ [substituting in (ii)]

$$\therefore \quad \tfrac{5}{2}=4.3-\tfrac{1}{2}.3^2+x_0 \quad \text{or} \quad x_0=-5.$$

$$\therefore \quad x=4t-\tfrac{1}{2}t^2-5. \qquad \text{.. (iii)}$$

In the range 6-7 s, $\frac{dv_x}{dt}=2$

or $\quad v_x=2t+v_0$ (after integration).

At $t=6$, $v_x=-2$. $\therefore$ $-2=2\times 6+v_0$ or $v_0=-14$. $\therefore$ $v_x=2t-14$

or $\quad \frac{dx}{dt}=2t-14$

or $\quad x=t^2-14t+x_0$ (after integration).

At $t=6$, $x=4\cdot 6-\frac{1}{2}.6^2-5=1$ [substituting in (iii)]

$$\therefore \quad 1=6^2-14\times 6+x_0 \quad \text{or} \quad x_0=49$$

$$\therefore \quad x=t^2-14t+49. \qquad \text{.. (iv)}$$

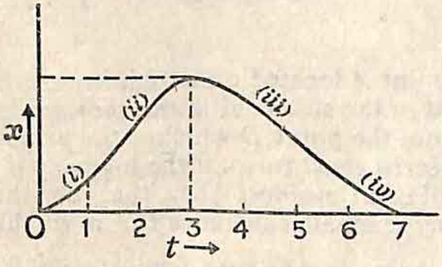


Fig. 1.12

The displacement at $t=7$ sec. is obtained from Eq. (iv)

$$x=7^2-14\times 7+49=0. \quad \textit{Ans.}$$

The distance covered in 7 sec. is given by the area of the v_x-t graph, and is (neglecting $-$ve sign)

$$S=\tfrac{1}{2}\times 1\times 1+2\times 1+\tfrac{1}{2}\times 1\times 1+\tfrac{1}{2}\times 2\times 2+\tfrac{1}{2}\times 1\times 2=6 \text{ unit.} \quad \textit{Ans.}$$

12. Three points are located at the vertices of an equilateral triangle whose sides equals a. They all start moving simultaneously with speed v, with the first point heading continually for the second, the second for third, and the third for the first. How soon will the points meet ?

Sol. The points will meet at the centroid of the triangle. Since each point keeps bearings towards the other point, each will be at the corners of gradually diminishing equilateral triangle (see figure), so they are all ultimately at the centroid of the original triangle. The velocity of each at any instant is along the tangent to the curves. The component of the velocity along the medians of the instantaneous equilateral triangle on which they lie is the effective velocity towards the centroid of the triangles. This component velocity is $v\cos30° = \frac{v\sqrt{3}}{2}$.

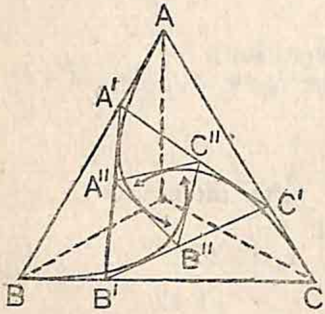


Fig. 1.13

The distance of each corner of the original triangle from the centroid $= \frac{a}{2}\sec30° = \frac{a}{\sqrt{3}}$.

Therefore, the time after which they meet $= \dfrac{\frac{a}{\sqrt{3}}}{\frac{v\sqrt{3}}{2}} = \frac{2}{3} \cdot \frac{a}{v}$. *Ans.*

13. From a point A located on a highway a motorist has to reach a point B located in the side field in the shortest possible time. At what distance from the point D which is the projection of B on the highway the motorist must turn off the highway if the speed of the motor in the field is n times less than that on the highway ? The point B is at a perpendicular distance l from the highway.

Sol. Let the motorist take turn from the point C at a distance x from A. Draw perpendicular BD from B on the highway.

Let $CD = x$ and $AB = a$. Then time of travel from A to B is given by

$$t = \frac{a-x}{v} + \frac{\sqrt{l^2+x^2}}{\frac{v}{n}} = \frac{a-x}{v} + \frac{n}{v}\sqrt{l^2+x^2}.$$

$$\therefore \quad \frac{dt}{dx}=-\frac{1}{v}+\frac{n}{v}\cdot\frac{1}{2}(l^2+x^2)^{-\frac{1}{2}}2x$$

$$=-\frac{1}{v}+\frac{nx}{v\sqrt{l^2+x^2}}.$$

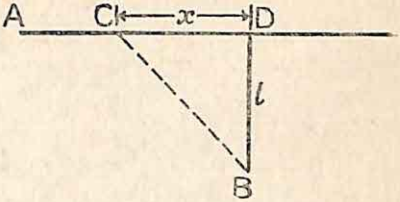


Fig. 1.14

When t is minimum, $\frac{dt}{dx}=0$.

$$\therefore \quad \frac{1}{v}=\frac{nx}{v\sqrt{l^2+x^2}}, \quad \text{or} \quad l^2+x^2=n^2x^2$$

or
$$x=\frac{l}{\sqrt{n^2-1}}. \text{ Ans.}$$

14. A point A moves uniformly with velocity v in such a way that the direction of its velocity continually points at another point B which in its turn moves along a straight line with uniform velocity u ($u<v$). At the initial moment u and v are at right angles and the points are separated by a distance l. How soon will the point meet ?

Sol. The point A moves towards B with the speed v at any instant. At the same time B moves away from A with the speed $u\cos\alpha$ where α is the inclination of the line AB because $u\cos\alpha$ is the component of the velocity of B along AB. Therefore, the distance AB reduces at the rate of $(v-u\cos\alpha)$. The initial separation between A and B was l. This separation is decreased at this rate and is reduced to zero when they (A and B) meet.

Let A and B meet after time τ. Then obviously

$$\int_0^\tau (v-u\cos\alpha)dt=l. \quad \text{.. (i)}$$

Along OX, i.e., the line along which B moves the distance described by B in time τ is $u\tau$. Parallel to OX the velocity of A is $v\cos\alpha$. Therefore, distance described by A along OX is $\int_0^\tau v\cos\alpha\, dt$ When they meet it is required that $\int_0^\tau v\cos\alpha\, dt$ be equal to $u\tau$.

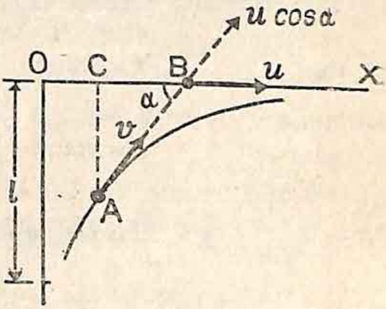


Fig. 1.15

$$\therefore \int_0^\tau v\cos\alpha\, dt = u\tau \quad \text{.. (ii)}$$

or
$$\int_0^\tau \cos\alpha\, dt = \frac{u\tau}{v} \quad \text{.. (iii).}$$

(We cannot take $\cos\alpha$ to be constant because α changes with time.)

From (i), we have $\int_0^\tau v\, dt - \int_0^\tau u\cos\alpha\, dt = l$

or
$$v\tau - u\int_0^\tau \cos\alpha\, dt = l.$$

Substituting the value of the integral from (iii), we have

$$v\tau - u.\frac{u\tau}{v} = l$$

or
$$\tau = \frac{lv}{v^2 - u^2}. \textit{ Ans.}$$

15. A shot is fired with a velocity u at a vertical wall whose distance from the point of projection is x. Prove that the greatest height above the level of the point of projection at which the bullet can hit the wall is $\frac{u^4 - g^2x^2}{2gu^2}$.

Sol. Let θ be the angle of projection. Applying kinematic equations to the horizontal and vertical motion of the shot, we have

$$x = u\cos\theta.t \quad \text{and} \quad h = u\sin\theta.t - \tfrac{1}{2}gt^2,$$

where h is the required height

or
$$h = u\sin\theta.\frac{x}{u\cos\theta} - \tfrac{1}{2}g.\frac{x^2}{u^2\cos^2\theta}$$

or
$$h = x\tan\theta - \tfrac{1}{2}g\frac{x^2}{u^2}\sec^2\theta$$

$$\therefore \frac{dh}{d\theta} = x\sec^2\theta - \tfrac{1}{2}g\frac{x^2}{u^2}2\sec\theta\sec\theta\tan\theta$$

$$= x\sec^2\theta - g\frac{x^2}{u^2}\sec^2\theta\tan\theta = x\sec^2\theta\left(1 - \frac{xg\tan\theta}{u^2}\right).$$

When h is maximum, $\frac{dh}{d\theta}=0$.

$$\therefore \quad 1-\frac{gx}{u^2}\tan\theta=0, \quad \text{or} \quad \tan\theta=\frac{u^2}{gx}.$$

$$\therefore \quad h_{max}=x.\frac{u^2}{gx}-\tfrac{1}{2}.\frac{gx^2}{u^2}\left(1+\frac{u^4}{x^2g^2}\right)=\frac{u^2}{g}-\frac{gx^2}{2u^2}-\frac{u^2}{2g}$$

$$\text{or} \quad h_{max}=\frac{2u^4-g^2x^2-u^4}{2u^2g}=\frac{u^4-g^2x^2}{2u^2g}. \textit{ Proved.}$$

EXERCISE

CRITICAL QUESTIONS

1. The acceleration vector of a particle is a constant. The trajectory of the particle is (a) parabola, (b) an ellipse, (c) hyperbola, (d) a circle.

2. The mod of acceleration vector of a particle is constant. The trajectory of the particle is (a) parabola, (b) an ellipse, (c) hyperbola, (d) a circle.

3. A coin is allowed to fall in a moving train. What is the trajectory of the coin in a frame fixed to the train and to the frame fixed to the earth ?

4. A black dot is made at the tip of an aerofoil of an aeroplane. What is the trajectory of the black dot as it appears to the pilot and to an observer on the ground ?

5. Can you suggest a method to measure (a) thickness of the page of this book, (b) radius of the nucleous of an atom.

6. What is the advantage of having atomic standards ?

7. The 'metre' is defined in terms of wavelength of light emitted by (a) krypton, (b) cadmium, (c) calcium, (d) neon.

8. Can a vector be zero if one of its components is not zero ?

9. Does a scalar quantity depend on the frame of reference chosen ?

10. We can order events in time such as past, present and future. Hence there is a sense of time. Is time a vector therefore ? If not, why not ?

11. Is the frame of reference attached to the earth strictly inertial or noninertial ?

12. Is the vector product of two polar vectors axial or polar ? Give an example.

13. The scalar product of two vectors is a scalar. True or false ?

14. Dimensional methods can be used to connect any number of physical quantities which are guessed to be interconnected. True or false ?

15. Average speed can mean the magnitude of the average velocity vector. Another meaning given to it is that the average speed is the total length of the path traversed divided by the elapsed time. Are these meanings different ? If so, give example.

16 Can a body have zero velocity and still be accelerating ? Can a body have a constant speed and still have a varying velocity ? Can a body have a constant velocity and still have a varying speed ?

17. Can an object have an eastward velocity while experiencing a westward acceleration ?

OBJECTIVES QUESTIONS

1. If in a system the force of attraction between two point masses each of 1 kg situated at 1 km apart is taken as a unit force and is called 'notwen' (newton written in reverse order), then if $G=6{\cdot}67\times10^{-11}\text{Nm}^2\text{kg}^{-2}$ in SI, the relation of newton and notwen is

(a) 1 notwen $=6{\cdot}67\times10^{-11}$ newton, (b) 1 newton $=6{\cdot}67\times10^{-17}$ notwen, (c) 1 notwen $=6{\cdot}67\times10^{-17}$ newton, (d) 1 newton $=6{\cdot}67\times10^{-12}$ notwen.

2. The dimensions of coefficient of viscosity is (a) ML^2T^{-2}, (b) $ML^{-1}T^{-1}$, (c) $M^2L^3T^{-2}$, (d) MLT^{-1}.

3. $ML^3T^{-1}Q^{-2}$ (Q is the dimension of charge) are the dimensions of (a) resistivity, (b) capacitance, (c) inductance, (d) potential difference.

4. L^2T^{-2} are the dimensions of (a) specific latent heat, (b) momentum, (c) thermal conductivity, (d) specific heat capacity.

5. The dimensions of ϵ_0 in terms of M, L, T, Q (dimensions of charge) are (a) $MLT^{-3}Q^{-2}$, (b) $M^{-1}L^{-1}T^{-3}\,Q^2$, (c) $M^{-1}L^{-3}T^2Q^2$, (d) $M^{-1}L^{-2}T^2Q^2$.

6. The dimensions of angular accelerations are (a) $M^1L^1T^{-2}$, (b) $M^0L^0T^{-2}$, (c) ML, (d) $ML^{-3}T^{-2}$.

7. The coefficient of thermal conductivity is (a) $MLT^{-3}K^{-1}$, (b) $M^2L^{-1}T^{+2}K$, (c) $MLT^{-2}K^{-1}$, (d) $MLT^{-3}K^{-2}$.

8. A man standing on the edge of a cliff at some height above the ground below throws one ball 1 straight up with initial speed u and then throws another ball 2 straight down with the same initial speed. (a) Ball 1 has larger speed than ball 2 when they hit the ground, (b) ball 2 has larger speed than ball 1 when they hit the ground, (c) both hit the ground with the same speed.

9. In the arrangement shown in the figure if v_1 and v_2 are instantaneous velocities of masses m_1 and m_2 respectively and angle $ACB = 2\theta$ at that instant then

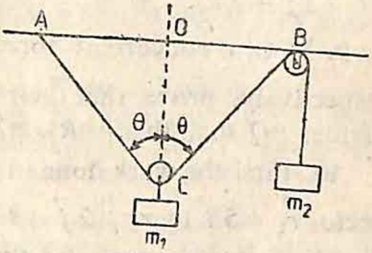


Fig. 1.16

(a) $\theta = \cos^{-1} \frac{v_1}{2v_2}$,

(b) $\theta = \cos^{-1} \frac{v_2}{2v_1}$,

(c) $\theta = \tan^{-1} \frac{v_1}{2v_2}$, (d) $\theta = \sin^{-1} \frac{v_1}{v_2}$.

NUMERICAL QUESTIONS

1. If in a system the force of attraction between two electrons 1 m apart in vacuum is taken as unit force (call it a femton) and the charge on an electron is taken as unit charge (call it couleon), calculate the value of ϵ_0 in this system. How many femtons equal a newton ? Given that the value of ϵ_0 in SI is $\frac{1}{36\pi \times 10^9}$ and the value of electron charge is $1{\cdot}6 \times 10^{-19}$ coulomb.

2. Suppose that a man defines a unit of force as that which acts due to gravitation between two point masses each of 1 kg and 1 m apart. What would be the value of G, the constant of gravitation, in this system ? What would be the value of a newton in this system ? (Given that G in SI $= 6{\cdot}6 \times 10^{-11}$) [JEE (W. Bengal) 1976]

3. Consider energy, length, time and charge as bases and denote them by U, L, T and Q. Find the dimensions of (a) mass, (b) ϵ_0 and (c) resistance.

4. Show by the method of dimensions that the energy per unit volume in an electric field of strength E is $u = \frac{1}{2}\epsilon E^2$.

5. Show by the method of vectors that the line joining the midpoint of the hypotenuse of a right-angled triangle to the 90°-corner is half the hypotenuse.

6. Show that the vectors $3\vec{i}+4\vec{j}-5\vec{k}$ and $\vec{i}-2\vec{j}-\vec{k}$ are at right angles.

7. Find the value of a so that $2\vec{i}+a\vec{j}+\vec{k}$ and $4\vec{i}-2\vec{j}-2\vec{k}$ are perpendicular.

8. Find the unit vector perpendicular to $2\vec{i}+2\vec{j}-\vec{k}$ and $6\vec{i}-3\vec{j}+2\vec{k}$.

9. If two concurrent forces be represented by $n\vec{OP}$ and $m\vec{OQ}$ respectively, prove that their resultant is given by $(m+n)\ \vec{OR}$ if R divides PQ such that $nPR=mRQ$.

10. Find the work done in moving a particle from the position vector $\vec{r_1}=5\vec{i}$ to $\vec{r_2}=2\vec{j}+4\vec{k}$ by a force $\vec{F}=2\vec{i}-\vec{j}-\vec{k}$.

11. A stone is projected from the surface of a flat field with a speed of 20 ms^{-1} and at an angle of 53·1° above the horizontal. Find the time of flight and the projectile range.
($\sin 53{\cdot}1^\circ=0{\cdot}8$ and $\cos 53{\cdot}1^\circ=0{\cdot}6$.) **(N. D. A. 1978)**

12. A ball is dropped from a balloon when it is 392 m above the ground and ascending at 9·8 ms^{-1}. Find the time taken by the ball to hit the ground. ($g=9{\cdot}8\ \text{ms}^{-2}$)

13. Water drops from the roof of a building 10 m high. The drops fall at regular intervals of time, the first drop striking the floor at the instant the fourth drop begins to fall. Find the positions of the individual drops when a drop strikes the floor.

14. A lead ball is dropped into a lake from a diving board 4·9 m above the water. It hits the water with a certain velocity and sinks to the bottom with the same velocity acquired by it on reaching the surface of water. It reaches the bottom 5 s after it is dropped. How deep is the lake ? Suppose all the water is drained from the lake. The ball is thrown from the diving board so that it again reaches the bottom in 5 s. What is the initial speed of the ball ?

15. An elevator car whose floor-to-ceiling distance is equal to 2·7 m starts ascending with constant acceleration 1·2 ms^{-2}. After 2 s a loose bolt gets released from its ceiling. Find the bolt's free fall time and the distance covered by the bolt during the free fall in the frame of reference fixed to the elevator shaft.

16. Two bodies were thrown simultaneously from the same position; one, straight up, and the other, at an angle of $\theta = 60°$ to the horizontal. The initial velocity of each body is equal $v = 25$ ms^{-1}. Neglecting the air drag, find the distance between the two bodies $t = 1{\cdot}7$ s later.

17. A point traversed half a circle of radius R during time interval τ. Calculate the modulus of the mean vector of the total acceleration if the point moved with constant tangential acceleration.

18. The position of a particle moving along the x-axis depends on the time according to the equation $x = (3t^2 - t^3)$ m. At what time does the particle reach its maximum x-position ? What total length of path does the particle cover in the first 4 s ? What is the displacement during the first 4 s ?

19. Each second a rabbit moves half the remaining distance from his nose to head of a lettuce. Does he ever get to the lettuce ? What is the limiting value of his average velocity ? Draw graphs showing his velocity and positions as the time increases.

20. The two ends of a train moving with a constant acceleration pass a certain point with velocities u and v. What is the velocity with which the middle point of the train passes the same point ?

21. A train moves from one station to another in two hours time. Its speed during the motion is shown in the graph (Fig. 1.17).

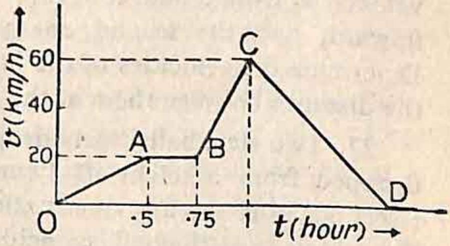


Fig. 1.17

Calculate : (i) maximum acceleration during the journey,
(ii) distance covered during the time interval from 0·75 to 1 hour.
(I. I. T. 1965)

22. The velocity of a particle varies with time as shown in the figure 1·18. Plot a graph showing the displacement of the particle with time. Draw also a graph showing variation of velocity with displacement. What is the total distance described by the particle in 8 s and what is the displacement at the end of 8 s ?

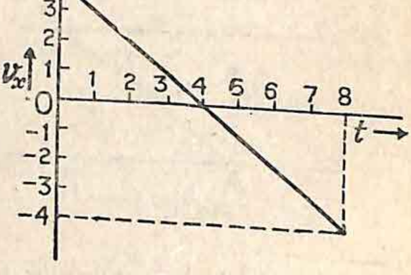


Fig. 1.18

23. A balloon starts rising from rest with acceleration 4 ms^{-2}. A stone is dropped from the balloon when it has gone up for 2 s. Calculate the time when it reaches the ground relative to an observer in the balloon and relative to an observer on the ground. Refer the time from the instant when the stone is dropped.

24. A motor boat going downstream overcame a raft at a point A. After $\tau=60$ min it turned back and passes the raft at a distance $l=6$ km from the point A. Find the velocity of flow of the river assuming the duty of the engine constant.

25. Two swimmers leave a point A on one bank of a river to reach a point B lying right across on the other bank. One of them crosses the river along the straight line AB while the other swims at right angles to the stream and then walks the distance that he has been carried away by the stream to get to the point B. What was the velocity u of walking if both swimmers reached simultaneously? The stream velocity $v_0=2$ kph and the velocity v' of each swimmer with respect to water equals 2·5 kph.

26. Two bodies are thrown one after the other with the same velocity v_0 from a high tower. The first body is thrown vertically upward, and the second one vertically downward after time τ. Determine the velocities of the bodies with respect to each other and the distance between them at the moment $t>\tau$.

27. Two steel balls freely drop on to an elastic plate*, the first is dropped from a height of 44 cm and the second from a height 11 cm τ seconds after the first one is allowed to fall. After a certain time the velocities of the balls coincide in magnitude and direction. Find the time τ and the interval during which the velocities of the two balls remain the same. The balls do not collide.

28. Two intersecting straight lines move out parallel to themselves with the speeds v_1 and v_2. Calculate the speed of the point of intersection of the lines if the angle between them is α.

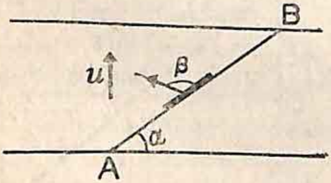


Fig. 1.19

29. A launch travels across a river from a point A to a point B on the opposite bank along the straight line AB forming an angle α with the bank. The flag on the mast of the launch makes an angle β with its direction of motion. Determine the speed of the launch with respect to the bank.

---

*Elastic plate means that the steel balls rebound with the same speed.

30. A man in a row boat must get from point A to point B on the opposite bank of the river (Fig. 1.20). The distance $BC = a$. The width of the river $AC = b$. At what minimum speed u relative to the still water should the boat travel to reach the point B? The velocity of flow of the river is v_0.

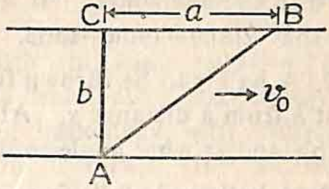


Fig. 1.20

31. A bus is running along a highway at a speed of $v_1 = 16$ ms^{-1}. A man is at distance $a = 60$ m from the highway and at a distance $b = 400$ m from the bus. In what direction should the man run to reach any point of the highway at the same time as the bus or before? The man can run at a speed of $v_2 = 4$ ms^{-1}. Given that in a '3 : 4 : 5 right-angled triangle' the angles are 37° and 53°.

32. The accelerator of a train can produce uniform acceleration $0{\cdot}25$ ms^{-2} and its brake can produce retardation $0{\cdot}5$ ms^{-2}. What is the shortest time in which the train can travel between two stations 8 km apart, if it stops at both stations? What is the maximum speed attained and for how long does the train move with uniform velocity?

33. Two ships A and B originally at a distance $a = 3$ km from each other depart at the same time from a straight coast line. Ship A moves along a straight line perpendicular to the shore while ship B constantly heads for ship A with the same speed v_0. After a sufficiently long interval B will obviously follow A maintaining a certain distance. Find this distance.

34. At what angle to the horizon should a stone be thrown from a steep bank of a river so that it may fall into the water as far as possible from the bank? The height of the bank is h and the velocity of projection is v. Use this result to find the angle of projection for maximum range in a flat horizontal plane.

35. Two bodies are thrown at the same time and with an equal velocity v_0 at angles α_1 and α_2 to the horizon. What is the velocity with which the bodies move relative to each other? What will the distance between the bodies be after time t elapses?

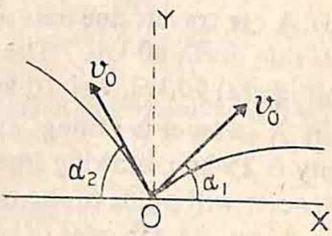


Fig. 1.21

36. A particle is projected from a point A at an angle α. It passes through a point B while travelling upwards in a direction making an angle β to the horizontal. If AB makes an angle θ to the horizontal show that $2\tan\theta=\tan\alpha+\tan\beta$.

37. A ball is to be thrown from the ground over a vertical wall of height h from a distance s. At what minimum initial velocity is this possible and at what angle to the horizon ?

38. A ball freely falls from a height h onto an inclined plane forming an angle α with the horizon. Find the ratio of the successive ranges of the ball along the plane. Consider the impacts between the ball and the plane to be absolutely elastic.

39. A dive-bomber moving at a speed $V=360$ kph at height $h=2$ km sights a target on the ground at a distance $l=\sqrt{5}$ km. from itself. At what angle to the horizon should it dive to hit the target ? ($g=10$ m/s^2).

40. The current velocity of a river grows in proportion to the distance from its bank and reaches its maximum value v_0 in the middle. Near the banks the velocity is zero. A boat is so moving on the river that it is always perpendicular to the current and the speed of the boat in still water is u. Find the distance through which the boat crossing the river will be carried away by the current if the width of the river is c. Also determine the trajectory of the boat.

TEST YOURSELF

1. (i) A river flowing from west to east at a speed of 5 m/min. A man on the south bank of the river, capable of swimming at 10 metres per minute in still water, wants to swim across the river in the shortest time. If the river is 1 km wide, the shortest time in which he can cross the river is (a) 1 h 50 min, (b) 3 h 20 min, (c) 1 h 40 min, (d) 2 h 30 min.

(ii) A car travels due east on a level road for 30 km. It then travels due north 40 km. The resultant vector displacement has a magnitude (a) 70 km, (b) 10 km, (c) 35 km, (d) 50 km.

(iii) A steamer is sailing directly south at 25 kph. The wind velocity is 25 kph, blowing from west to east. A flag on the mast of the steamer will point to the direction (a) 45° east of north (b) 45° south of east, (c) 45° west of north, (d) 45° west of south to an observer on the deck of the steamer.

(iv) If a journey is made at a steady speed of 36 kph and the return trip over the same route at steady speed of 45 kph, the average speed for the round trip is (a) 40·5 kph, (b) 18/5 kph, (c) 40 kph, (d) 36 kph.

(v) If two particles are projected with the same velocity, one vertically upward and the other vertically downward from the top of a tower (a) they hit the ground simultaneously, with different velocities, (b) they hit the ground simultaneously with the same velocity, (c) they hit the ground at different instants with different velocities, (d) they hit the ground on different instants with the same velocity.

(vi) The dimensions of impulse of a force is

(a) MLT^{-2}, (b) MLT^{-1}, (c) $ML^{-1}T^{-1}$, (d) $M^0L^{-1}T$.

(vii) If newton is redefined as the force of attraction between two unit masses (each of 1 kg) 1 metre apart, the value of G is (a) 10 newton $\text{kg}^{-2}\,\text{m}^2$, (b) 1 newton $\text{kg}^{-2}\,\text{m}^{-2}$, (c) 1 newton kg^{-2}m^2, (d) 100 newton $\text{kg}^{-2}\,\text{m}^2$.

(viii) A stone is dropped from a balloon rising with acceleration a, the acceleration of the stone relative to the balloon is (a) g downward, (b) $g+a$ downward, (c) $g-a$ upward, (d) $g+a$ upward, (e) $g-a$ downward.

(ix) Two lines AB and CD intersect at O at inclination α. If they move out parallel to themselves with speed v, the speed of O is (a) v, (b) $2v$, (c) $v\cos\alpha$, (d) $v\,\text{cosec}\,\alpha/2$.

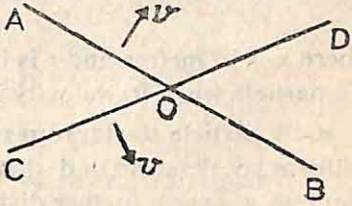


Fig. 1.22

(x) A point traverses quarter of a circle of radius R during the time interval τ. The modulus of the average velocity vector during the interval is (a) zero, (b) $\frac{2R}{\tau}$, (c) $\frac{\sqrt{2}R}{\tau}$, (d) $\frac{\pi R}{4\tau}$.

marks $3x10=30$

2. Answer the following questions :

(i) Match the physical quantities given in column I with dimensions in column II. $5\times2=10$

I	II
Angular momentum	$MLT^{-3}K^{-1}$
Torque	$ML^{-1}T^{-2}$

I	II
Pressure	ML^2T^{-2}
Conductivity	$ML^2T^{-2}Q^{-1}$
Electromotive force	ML^2T^{-1}

(ii) Fill up the gaps—

(a) A second is the time taken by 9192631770 cycles of the radiation from the hyperfine transition of.....when unperturbed by external fields. 1

(b) A metre is defined as 1650763·73 of wavelengths, in vacuo of the orange light emitted by......in transition from....to..... 2

(c) An area can be treated as a vector of modulus equal to the....and of direction along the..... 2

(iii) (a) Angular velocity is a polar vector. True or false ? 1

(b) Two balls of different masses are thrown vertically upward with the same speed. They pass through the point of projection in their downward motion with the same speed. True or false ? Neglect air resistance. 2

(c) Dimensions are very useful in checking the final results of a problem. True or false ? 2

Answer any five of the following. $5\times10=50$

3. The displacement of a particle moving in one dimension under the action of a force is related to the time by the equation

$$t=\sqrt{x}+3$$

where x is in metres and t is in seconds. Find the displacement of the particle when its velocity is zero. **(IIT 1979)**

4. A particle starts from rest and traverses a distance 's' with a uniform acceleration and then moves uniformly with the velocity acquired v_0 over a further distance '$2s$'. Finally it comes to rest after moving through a distance '$3s$' with uniform retardation. Assuming that the entire path is a straight line, find the average speed over the entire journey.

5. A ball falls under gravity from a height of 10 m, with an initial downward velocity v_0. It collides with the ground, loses 50% of its energy in collision, and then rises back to the same height. Find (i) the initial velocity v_0 and (ii) the height to which the ball would rise, after collision, if the initial velocity v_0 was directed upward, instead of downward. **(IIT 1979)**

6. A particle with an initial velocity v_0 in a plane is subjected to a constant acceleration in the same plane. Show that in general the path of the particle is a parabola. **(BIT, Mesra 1983)**

7. A rocket is fired vertically from the ground with a resultant vertical acceleration of 10 ms^{-2}. The fuel is finished in 1 minute and it continues to move up. What is the maximum height reached ? (IIT 1975)

8. The velocity of a particle moving in the positive direction of the x-axis varies as $v=\alpha\sqrt{x}$, where α is a positive constant. Assuming at $t=0$, $v=0$, find

(a) the time dependence of the velocity and the acceleration, (b) the mean velocity of the particle averaged over the time that the particle takes to cover the first s metres of the path.

9. (a) A point moves in the xy plane according to the law $x=a\sin\omega t$, $y=a(1-\cos\omega t)$, where a and ω are positive constants. Find the distance traversed by the point during the time τ.

(b) A man stands on the steep shore of a lake and pulls a boat in the water by a rope which he takes up at a constant speed v. What will the speed of the boat be at the moment when the angle between the rope and the surface of the water is α ?

Answers of Exercises

For critical questions and objectives, see solutions.

1. $4{\cdot}3\times10^{27}$. 2. 1 notwen $\times m^2\times kg^{-2}$; 1 newtons $1{\cdot}5\times10^{10}$ notwen.

3. (a) $UL^{-2}T^2$, (b) $U^{-1}L^{-1}Q^2$, (c) UTQ^{-2}.

7. $a=3$, 8. $\dfrac{\vec{i}-10\vec{j}-18\vec{k}}{5\sqrt{17}}$. 10. -16 units,

11. 3·265; 39·12 m. 12. 10 s.

13. 0, 1·11 m, 4·44 m and 16 m. 14. 49 m; 13·7 m/s.

15. ·7 s; ·72 m. 16. 21·6 m. 17. $\dfrac{2\pi R}{\tau^2}$.

18. 2 s; 24 m; -16 m. 19. see solution.

20. $\sqrt{\dfrac{u^2+v^2}{2}}$. 21. 160 km/h^2; 10 km. 22. 16 m; 0.

23. 2·3 s; 2·3 s. 24. 3 km/h. 25. 3 km/h.

26. $2v_0-g\tau$, $(2v_0-g\tau)t-v_0\tau+\frac{1}{2}g\tau^2$. 27. ·3 s.

28. $\sqrt{v_1{}^2+v_2{}^2+2v_1v_2\cos\alpha}\ \operatorname{cosec}\alpha$.

29. $\dfrac{u\sin(\alpha+\beta-\pi/2)}{\sin\beta}$. 30. $u_0=\dfrac{v_0b}{\sqrt{a^2+b^2}}$.

31. see the solution. **32.** 5 min 10 s; 186 km/h.

33. $a/2$. **34.** $\tan^{-1}\frac{v}{\sqrt{v^2+2gh}}$, 45°. **35.** $2v_0\tau\cos\frac{\alpha_1+\alpha_2}{2}$.

37. $\frac{\pi}{2}-\frac{1}{2}\tan^{-1}\frac{S}{h}$, $u_{min}=\sqrt{g(h+\sqrt{h^2+s^2})}$.

38. $S_1 : S_2 : S_3 \ldots = 1 : 2 : 3 \ldots$.

39. 45°. **40.** $\frac{1}{2}\,\frac{cv}{u}$.

Answers of Test Questions

1. (i). c. (ii). d, (iii). a, (iv). c, (v). d, (vi). b, (vii). c, (viii). b, (ix). d, (x). c.

2. see solution. **3.** 0. **4.** $\frac{3}{5}v_0$. **5.** 14 m/s.

7. 36·4 km. **8.** (a) $\frac{1}{2}\alpha^2 t$; $\frac{1}{2}\alpha^2$; (b) $\frac{\alpha\sqrt{S}}{2}$.

9. (a) $a\omega\tau$, (b) $\frac{v}{\cos\alpha}$.

Solutions and Hints to Exercises

CRITICAL QUESTIONS

1. (a). Example, a body falling freely under gravity.

2. (d).

3. A vertical straight line path relative to the frame fixed to the train and a parabolic path relative to the frame fixed to the earth.

4. A circular path relative to the pilot and a helical path relative to the observer fixed to the earth.

5. (a) Micrometer screw gauge, (b) α-ray scattering.

6. The atomic standards have the advantage of 'invariability' with respect to time and space.

7. (a).

8. A vector cannot be zero if one of its components is not zero.

9. No. A scalar quantity does not depend on the frame of reference.

10. Time is not a vector though it may have a sense because in order to be a vector it must add and subtract by vector algebra rules.

The times do not add or subtract by vector algebra rules. So it is not a vector.

11. It is noninertial because the earth is rotating about its axis. So a frame attached to its surface will have centripetal acceleration and it is an accelerated, that is, noninertial frame.

12. Axial, for example $\vec{\omega} = \vec{r} \times \vec{v}$.

13. True.

14. False. Only four quantities can be connected by the method of dimensions.

15. The meanings are different. The point can be made clear by considering the motion of a particle along a circle. Let a particle describe half of a circle of radius R in time τ.

$$<\vec{v}> = \text{average velocity vector} = \frac{-R\vec{j} - R\,\vec{i}}{\tau} = -\frac{2R}{\tau}\vec{i}$$

$$\therefore \quad |<\vec{v}>| = \frac{2R}{\tau}.$$

$$<v> = \text{average speed} = \frac{\pi R}{\tau}.$$

Obviously $<v> \neq |<\vec{v}>|$, that is, average speed as the modulus of the average velocity vector is not the same as the average speed defined as the total length of the path traversed divided by the elapsed time.

16. Yes, for example a particle in SHM at the extreme position has zero velocity but its acceleration is maximum.

Yes, a particle in circular motion has constant speed but varying velocity. No, a body cannot have a constant velocity but still having a varying speed.

17. No, a body cannot have eastward velocity and westward acceleration.

OBJECTIVE QUESTIONS

1. (c). It is like this : $1 \text{ notwen} = G\,\frac{1 \text{ kg} \times 1 \text{ kg}}{1 \text{ km}^2}$

or $$G = 10^6 \text{ notwen m}^2 \text{ kg}^{-2}$$
$$= 6{\cdot}67 \times 10^{-11} \text{ newton m}^2 \text{ kg}^{-2}$$

$\therefore$ 10^6 notwen $= 6{\cdot}67 \times 10^{-11}$ newton or 1 notwen $=6{\cdot}67\times10^{-17}$ newton,

So the answer is (c).

2. (b).
3. (a).
4. (a).
5. (c).
6, (b).
7. (a).

8. (c) The two bodies will start their downward journey from the edge of the cliff with the same speed and so they hit the ground with the same speed.

9. (b). It is like this : Let $OC=x$ and the length of the string from B to m_2 is y and $AB=2a$.

Then $2\sqrt{a^2+x^2}+y=$ a constant because length of the string is fixed.

Differentiating, $2\times\frac{1}{2}(a^2+x^2)^{-\frac{1}{2}}2x\dot{x}+\dot{y}=0$

or $\dfrac{2x}{\sqrt{a^2+x^2}}\cdot v_1+v_2=0$ $\quad(\because\ \dot{x}=v_1$ and $\dot{y}=v_2)$

or $2\cos\theta v_1=-v_2$.

The minus sign shows that v_1 is opposite to v_2.

$\therefore\ 2\cos\theta\times v_1=v_2$ (numerically)

or $\cos\theta=\dfrac{v_2}{2v_1}$.

NUMERICAL QUESTIONS

1. The force of attraction is given by

$$\triangle F=\frac{1}{4\pi\epsilon_0}\ \frac{e\times e}{r^2}.$$

$$\therefore\ 1 \text{ femton}=\frac{1}{4\pi\epsilon_0}\times\frac{1 \text{ couleon}\times 1 \text{ couleon}}{(1\text{ m})^2}.$$

$$\therefore\ \epsilon_0=\frac{1}{4\pi}\text{ couleon}^2\text{ m}^{-2}\text{ femton}^{-1}\quad \textit{Ans.}$$

In SI, $\triangle F=9\times10^9\times\dfrac{(1{\cdot}6\times12^{-19})^2}{(1\text{ m})^2}=9\times2{\cdot}56\times10^{-29}$ newton

$\therefore$ 1 femton $=9\times2{\cdot}56\times10^{-29}$ newton or $1 \text{ newton} = \frac{1}{9 \times 2{\cdot}56 \times 10^{-29}} = 4{\cdot}3 \times 10^{27}$ femton. *Ans.*

2. Let us call the new unit of force 'notwen' then

$$1 \text{ notwen} = G\,\frac{1\text{ kg} \times 1\text{ kg}}{(1\text{ m})^2}$$

or $G = 1 \text{ notwen} \times \text{m}^2 \times \text{kg}^{-2}$. *Ans.*

In SI, $\triangle F = 6{\cdot}6 \times 10^{-11} \frac{1 \times 1}{1^2}$ newton

$\therefore$ $6{\cdot}6 \times 10^{-11}$ newton $= 1$ notwen

or $1 \text{ newton} = 1{\cdot}5 \times 10^{10}$ notwen. *Ans.*

3. (a) We have, energy = force × distance

$$\therefore \quad [\text{force}] = \frac{[\text{energy}]}{[\text{distance}]} = UL^{-1}.$$

Now force = mass × acceleration

$\therefore$ [force] = [mass] × [acceleration]

or $UL^{-1} = [\text{mass}] \times LT^{-2}$

$\therefore$ $[\text{mass}] = UL^{-2}T^2$. *Ans.*

(b) We have, $\triangle F = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2}$.

$$[\epsilon_0] = \frac{QQ}{UL^{-1}L^2} = U^{-1}L^{-1}Q^2. \quad \textit{Ans.}$$

(c) We have, $H = I^2Rt$

(Heat produced = current² × resistance × time)

$$\therefore \quad [R] = \frac{[H]}{[I^2][t]} = \frac{U}{Q^2T^{-2}T} \qquad \left(\because \quad \text{current} = \frac{\text{charge}}{\text{time}}\right)$$

$$= UTQ^{-2}. \quad \textit{Ans.}$$

4. Suppose $u = k\epsilon^a E^b$.

Let us take energy, length, time and charge as base quantities.

Dimension of $u = \frac{\text{energy}}{\text{volume}} = UL^{-3}$.

Dimension of $\epsilon = U^{-1}L^{-1}Q^2$.

Dimension of $E = \frac{\text{potential difference}}{\text{distance}} = \frac{UQ^{-1}}{L} = UL^{-1}Q^{-1}$

$$\left(\because \quad \text{potential difference} = \frac{\text{work done}}{\text{charge}}\right)$$

$$\therefore \quad UL^{-3}=(U^{-1}L^{-1}Q^{2})^{a}(UL^{-1}Q^{-1})^{b}=U^{-a+b}L^{-a-b}Q^{2a-b}.$$

$\therefore \quad -a+b=1;\ -a-b=-3$ and $2a-b=0$

or $\quad b=1+a;\ a+b=3$ and $2a=b$

or $\qquad a=1,\ b=2.$

$$\therefore \quad u=k\epsilon' E^{2}=k\epsilon E^{2}.$$

Experimentally, $\quad k=\frac{1}{2}.$

$$\therefore \quad u=\tfrac{1}{2}\epsilon E^{2}.$$

5. Let ABC be the triangle right angled at A. Take A as origin and AB and AC as x and y axis.

Let a and b be the lengths of AB and AC.

Then $AC=\sqrt{a^{2}+b^{2}}$ and the co-ordinates of B and C are $(a, 0)$ and $(0, b)$. Therefore, co-ordinates of the mid point M are

$$\left(\frac{a}{2}, \frac{b}{2}\right).$$

$$\therefore \quad \overrightarrow{AM}=\frac{a}{2}\vec{i}+\frac{b}{2}\vec{j}.$$

$$\therefore \quad AM=\sqrt{\frac{a^{2}}{4}+\frac{b^{2}}{4}}=\frac{1}{2}\sqrt{a^{2}+b^{2}}=\tfrac{1}{2}AC. \quad \textit{Proved.}$$

6. Let us consider 'dot product' of the vectors

$$(3\vec{i}+4\vec{j}-5\vec{k}).(\vec{i}-2\vec{j}-\vec{k})=3-8+5=0$$

$$\therefore \quad \cos\theta=0 \quad (\because \ \vec{a}.\vec{b}=ab\cos\theta)$$

or $\qquad \theta=\dfrac{\pi}{2}.$ *Proved.*

7. Let us consider dot product of the two vectors. Since the two vectors are at right angles, $\cos\theta=0$.

$$\therefore \quad (2\vec{i}+a\vec{j}+\vec{k}).(4\vec{i}-2\vec{j}-2\vec{k})=0.$$

$$8-2a-2=0, \text{ or } a=3. \quad \textit{Ans.}$$

8. The unit vector along the cross-product is the unit vector perp. to the given vector. Let $\vec{a}$ be their cross-product. Then

$$\vec{a}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & -1 \\ 6 & -3 & 2 \end{vmatrix}=\vec{i}-10\vec{j}-18\vec{k}.$$

$$\hat{a}=\frac{\vec{a}}{a}=\frac{\vec{i}-10\vec{j}-18\vec{k}}{\sqrt{1^2+10^2+18^2}}=\frac{\vec{i}-10\vec{j}-18\vec{k}}{5\sqrt{17}}.\quad Ans.$$

9. By triangle law of vector addition,

$$\vec{OP}=\vec{OR}+\vec{RP} \quad \text{or} \quad n\vec{OP}=n\vec{OR}+n\vec{RP}$$

and $$\vec{OQ}=\vec{OR}+\vec{RQ} \quad \text{or} \quad m\vec{OQ}=m\vec{OR}+m\vec{RQ}.$$

Adding, $$n\vec{OP}+m\vec{OQ}=(m+n)\vec{OR}+n\vec{RP}+m\vec{RQ}.$$

$$\because \quad n\vec{OP}+m\vec{OQ}=(m+n)\vec{OR} \text{ (given)},$$

$$n\vec{RP}+m\vec{RQ}=0$$

or $nRP\,\hat{a}+mRQ(-\hat{a})=0$ where $\hat{a}$ is the unit vector along PQ

or $$nRP=mRQ$$

or $$\frac{RP}{RQ}=\frac{m}{n}.\quad \textit{Proved.}$$

10. Work done is given by

$$W=\vec{F}\,.\,\vec{S}.$$

Here, $$\vec{S}=\vec{r_2}-\vec{r_1}=(2\vec{j}+4\vec{k})-5\vec{i}.$$

$$\therefore \quad W=(2\vec{i}-\vec{j}-\vec{k})\,.\,(-5\vec{i}+2\vec{j}+4\vec{k})$$
$$=-16 \text{ units.}\quad Ans.$$

11. Considering horizontal and vertical motion we have

$$x=20\cos 53{\cdot}1\times t=12t$$

and $$0=20\sin 33{\cdot}1\times t-\tfrac{1}{2}gt^2 \quad \text{or} \quad 16=\tfrac{1}{2}gt$$

or $$t=\frac{32}{9{\cdot}8}=3{\cdot}26 \text{ s.}\quad Ans.$$

and $$x=12\times 3{\cdot}26=39{\cdot}12 \text{ m.}\quad Ans.$$

12. Fix up a frame at a height 392 m above the earth with horizontal direction as x-axis and downward vertical as y-axis.

$$\therefore \quad 392=(-9{\cdot}8)t+\tfrac{1}{2}\times 9{\cdot}8t^2 \quad \text{or} \quad t^2-2t=80;$$

$$t=10 \text{ s.}\quad Ans.$$

13. Let t be the time to reach the ground.

Then $10=0.t+\frac{1}{2}\times 9{\cdot}8\times t^2$, or $t=\frac{10}{7}$ s.

If τ is the interval between two successive drops then

$$3\tau=\tfrac{10}{7} \quad \text{or} \quad \tau=\tfrac{10}{21} \text{ s.}$$

Let h_1 and h_2 be the distances of the third and second drop from the roof. Then

$$h_1=\tfrac{1}{2}\times9\cdot8\times(\tfrac{10}{21})^2=1\cdot11\text{ m}$$

and
$$h_2=\tfrac{1}{2}\times9\cdot8(\tfrac{20}{21})^2=4\cdot44\text{ m.}$$

So the drops are at distance 0, 1·11 m, 4·44 m and 10 m from the roof. *Ans.*

14. We have
$$v^2-0=2\times9\cdot8\times4\cdot9$$
or
$$v=2\times4\cdot9=9\cdot8\text{ m/s.}$$

$\therefore$ Depth of the lake $=9\cdot8\times5=49\cdot0$ m. *Ans.*

Total height $=4\cdot9+49\cdot0=53\cdot9$ m.

$$\therefore\quad 53\cdot9=v_0\times5+\tfrac{1}{2}\times9\cdot8\times5^2$$
$$v_0=-13\cdot7\text{ m/s.}\quad \textit{Ans.}$$

The minus sign shows that it is to be thrown upwards.

15. The velocity of the car and bolt after 2 s
$$=0\times2+\tfrac{1}{2}\times1\cdot2\times2^2=2\cdot4\text{ m/s. (upward)}$$

Initial relative velocity of the bolt $=0$.

Relative acceleration of the bolt $=g-(-a)=g+a=9\cdot8+1\cdot2$
$=11\cdot0\text{ m/s}^2$. (downward)

Relative distance covered $=2\cdot7$ m (downward).

$$\therefore\quad 2\cdot7=0\times t+\tfrac{1}{2}\times11\times t^2,\quad\text{or}\quad t=\cdot7\text{ s.}\quad \textit{Ans.}$$

The absolute velocity of the bolt $=2\cdot4$ m/s. (upward)

Absolute acceleration $=g=9\cdot8\text{ m/s}^2$. (downward)

$$\therefore\quad h=(-2\cdot4)\times\cdot7+\tfrac{1}{2}\times9\cdot8\times\cdot7^2=\cdot72\text{ m.}\quad \textit{Ans.}$$

16. Let us fiix up a frame of reference fixed to the earth.

Obviously, $x_1=0$

$$y_1=25\times1\cdot7+\tfrac{1}{2}(-9\cdot8)\times1\cdot7^2=42\cdot5-14\cdot2=28\cdot3\text{ m.}$$

For the second particle

$$x_2=25\cos60^\circ\times1\cdot5-25\times\cdot5\times1\cdot7=21\cdot3\text{ m.}$$
$$y_2=25\sin60^\circ\times1\cdot7+\tfrac{1}{2}(-9\cdot8)\times1\cdot7^2=24\cdot6\text{ m.}$$

Now $S=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=\sqrt{21\cdot3^2+(24\cdot6-28\cdot3)^2}$

or $S=\sqrt{453\cdot7+13\cdot7}=21\cdot6$ m. *Ans.*

17. Let v_i and v_f be the final and initial velocity modulus of the particle. Then

$$<\vec{a}>\text{ (average total acceleration vector)}=\frac{\vec{v}_f-\vec{v}_i}{\tau}$$

$$=\frac{v_f(-\vec{j})-v_i\,\vec{j}}{\tau}$$

or $$< \vec{a} > = -\frac{(v_i+v_f)}{\tau}\,\vec{j}$$

$$\therefore \quad | < \vec{a} > | = \frac{v_i+v_f}{\tau}.$$

Since the tangential acceleration is a constant, velocity increases uniformly from v_i to v_f. Hence the average speed during the interval

$$\frac{v_i+v_f}{2} = \frac{\pi R}{\tau}$$

or $$v_i+v_f = \frac{2\pi R}{\tau}$$

$$\therefore \quad | < \vec{a} > | = \frac{2\pi R}{\tau^2}. \quad \textit{Ans.}$$

18. We have $$x = 3t^2 - t^3.$$

$$\therefore \quad \frac{dx}{dt} = 6t - 3t^2$$

x is maximum when $$\frac{dx}{dt} = 0$$

$$\therefore \quad 6t - 3t^2 = 0 \quad \text{or} \quad t = 2 \text{ s.} \quad \textit{Ans.}$$

$$x_{max} = 3.2^2 - 2^3 = 12 - 8 = 4 \text{ m.}$$

So in the first 2 seconds the pariticle covers 4 m in the +ve direction, that is, to the right.

At $t=4$, $x = 3.4^2 - 4^3 = 48 - 64 = -16$ m.

Thus in the next 2 seconds covers 4 m to the left from the extreme right position and comes to the initial position and goes to the left of the origin by 16 m.

$\therefore$ Total distance covered $= 4+4+16 = 24$ m. *Ans.*

The displacement of the particle is given by x.

$\therefore$ x in 4 s $= -16$ m. *Ans.*

19. He will never reach the lettuce. Let us take the lettuce as the reference and let x_0 be the initial distance of the rabbit from the lettuce. After 1 second it is at a distance $x = \frac{x_0}{2}$, after 2 seconds it is at a distance $x = \frac{x_0}{4}$, after 3 seconds it is at distance $x = \frac{x_0}{8}$ and so on.

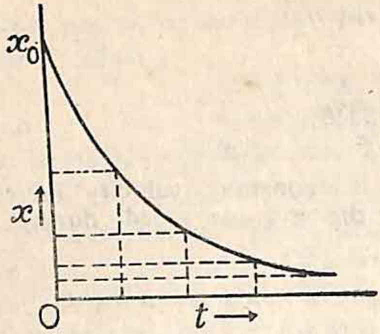


Fig. 1.22

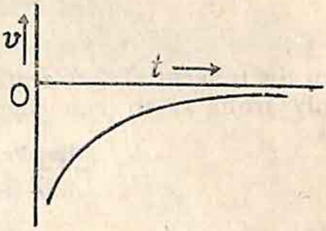


Fig 1.23

The velocity of a body is the slope of x-t plot, that is, $\frac{dx}{dt}=v$. It is seen clearly from x-t plot that distance of the rabbit from the lettuce is never zero.

20. Since the train is a rigid body, every point of it possesses the same instantaneous velocity and so we may say that a particle has its velocity increased from u to v with a distance equal to the length 'l' of the train.

$$\therefore \quad v^2-u^2=2al.$$

Let the same particle have its velocity increased from u to v' within $\frac{l}{2}$.

$$\therefore \quad v'^2-u^2=2.a\frac{l}{2}.$$

Eliminating a and b we have

$$\frac{v_1{}^2-u^2}{v'^2-u^2}=2, \quad \text{or} \quad v'=\sqrt{\frac{u^2+v^2}{2}}. \quad \textit{Ans.}$$

21. The maximum acceleration is the maximum slope. The BC part of the plot has got the maximum slope and hence its slope gives the maximum acceleration.

$$\therefore \quad \text{Maximum acceleration}=\frac{60-20}{1-{\cdot}75}=\frac{40}{{\cdot}25}=160 \text{ km/h}^2. \quad \textit{Ans.}$$

Distance covered = area of the plot from 0·75 to 1 hour

$= \frac{1}{2}(20+60)\times{\cdot}25$ [Formula : area of trapezium $=\frac{1}{2}(a+b)h$]

$=10$ km. *Ans.*

22. We have $a_x = \frac{dv_x}{dt} = -\frac{4}{4} = -1$ (Negative because the inclination of the line is greater than 90° and hence slope which is $\tan\theta$ is $-$ve)

or $\quad dv_x = -dt$, or $v_x = -t + v_0$ (say).

At $t=0$, $v_x = 4$ m/s; $\therefore$ $4 = v_0$.

$\therefore \quad v_x = 4 - t$.

$$\therefore \quad \frac{dx}{dt} = 4 - t$$

or $\quad x = 4t - \frac{1}{2}t^2 + x_0$.

At $t=0$, $x=0$. $\therefore$ $x_0 = 0$.

$\therefore \quad x = 4t - \frac{1}{2}t^2$.

This equation gives variation of displacement with time.

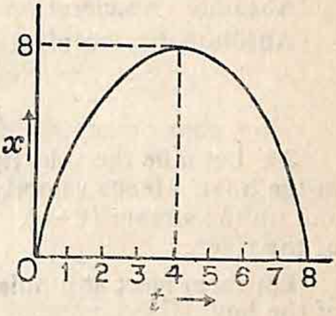


Fig. 1.24

$$\text{Now,} \quad \frac{dv_x}{dt} = -1, \quad \text{or} \quad \frac{dv_x}{dx} \cdot \frac{dx}{dt} = -1$$

$$\text{or} \quad v_x \frac{dv_x}{dx} = -1 \quad \left(\because \frac{dx}{dt} = v_x\right)$$

or $\quad v_x dv_x = -dx$.

Integrating, $\frac{1}{2}v_x^2 = -x + c'$ (a constant),

or $\quad v_x^2 = -2x + c$ (say).

When $x=0$, $v_x = 4$;

$\therefore \quad 16 = c$.

$\therefore \quad v_x^2 = 16 - 2x$.

This equation gives variation of velocity with distance.

The total distance covered = 8 m (forward) + 8 m (backward) = 16 m. *Ans.*

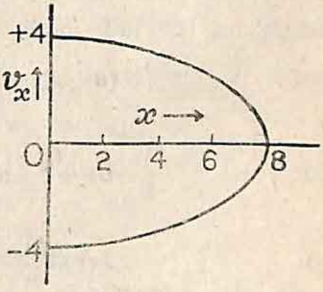


Fig. 1.25

Displacement at the end of 8 s $= 4 \times 8 - \frac{1}{2} \times 8^2 = 0$. *Ans.*

23. The height of the balloon at the time of dropping the stone
$= 0 \times 2 + \frac{1}{2} \times 4 \times 2^2 = 8$ m.

Velocity of the balloon $= 0 + 4.2 = 8$ m/s (upward).

Relative inital velocity of the stone $= 8 - 8 = 0$.

Relative acceleration of the stone $= g - (-a) = g + a = 9{\cdot}8 + 4$
$= 13{\cdot}8$ m/s^2 (downward).

Relative displacement $=8+(8t+2t^2)$.

$$\therefore \quad 8+8t+2t^2=0\times t+\tfrac{1}{2}\times 13{\cdot}8t^2$$

or $$4{\cdot}9t^2-8t-8=0, \quad \text{or} \quad t=2{\cdot}3 \text{ s.} \quad \textit{Ans.}$$

Absolute initial velocity (that is, velocity relative to the ground) $=8$ m/s (upward).

Absolute acceleration $=9{\cdot}8$ m/s^2 (downward).

Absolute displacement $=8$ m.

$$8=-8t+\tfrac{1}{2}\times 9{\cdot}8t^2$$

or $$t=2{\cdot}3 \text{ s.} \quad \textit{Ans.}$$

24. Let u be the velocity of the river water and v be the velocity of the boat. Hence velocity of the boat down the stream is $(u+v)$ and up the stream $(v-u)$. The velocity of the raft is same as that of the river.

Let them meet at a point B beyond A and O is the turning point of the boat.

Then $$AO=(u+v)\tau$$

$AB=l=u(t+\tau)$ where t is the time of travel of the boat from O to B.

and $$OB=(v-u)t=(v-u)\left(\frac{l}{u}-\tau\right)=\frac{lv}{u}-l-\tau(v-u).$$

But $$AO=AB+BO=l+\frac{lv}{u}-l-\tau(v-u)$$

or $$(u+v)\tau=\frac{lv}{u}-\tau(v-u)$$

or $$\tau(u+v+u-u)=\frac{lv}{u}$$

or $$2v\tau=\frac{lv}{u}$$

or $$u=\frac{l}{2\tau}=\frac{6 \text{ km}}{2\times 1 \text{ h}}=3 \text{ km/h.} \quad \textit{Ans.}$$

25. Let the first swimmer start at angle θ with the direction perpendicular to the river.

Then $$\sin\theta=\frac{v_0}{v'}.$$

The effective velocity along $AB=v'\cos\theta=\sqrt{v'^2-v_0^2}$.

$\therefore$ Time to cross the river by the first swimmer $= \dfrac{l}{\sqrt{v'^2 - v_0^2}}$

where l is the width of the river.

Time to cross the river by the second swimmer $= \dfrac{l}{v'}$

The distance by which he is drifted beyond $B = v_0 \times \dfrac{l}{v'}$.

If u is the speed of walking of the second swimmer, then time to walk back to B along the other bank is $\dfrac{v_0 l}{v'} \Big/ u$.

Since they reach simultaneously,

$$\frac{l}{\sqrt{v'^2 - v_0^2}} = \frac{l}{v'} + \frac{v_0 l}{v' u}, \quad \text{or} \quad \frac{v'}{\sqrt{v'^2 - v_0^2}} = 1 + \frac{v_0}{u}$$

$$\text{or} \quad \left(1 - \frac{v_0^2}{v'^2}\right)^{-\frac{1}{2}} = 1 + \frac{v_0}{u}$$

$$\text{or} \quad u = \frac{v_0}{\left(1 - \frac{v_0^2}{v'^2}\right)^{-\frac{1}{2}} - 1} = \frac{2}{\left(1 - \frac{2^2}{2 \cdot 5^2}\right)^{-\frac{1}{2}} - 1} = 3 \text{ km/h.} \quad \textit{Ans.}$$

26. The time of flight of the first is t and that of the second is $(t - \tau)$.

$$v_1 = v_0 - gt \quad \text{(upward)}$$
$$v_2 = v_0 + g(t - \tau) \quad \text{(downward).}$$

$\therefore$ v_{12} (relative velocity of 1 relative to 2) $= v_1 - (-v_2) = v_1 + v_2$

or $v_{12} = 2v_0 - g\tau$. *Ans.*

Now $x_1 = v_0 t - \frac{1}{2} g t^2$ (upward).

$x_2 = v_0(t - \tau) + \frac{1}{2} g (t - \tau)^2$ (downward).

$\therefore$ $x_{12} = x_1 - (-x_2) = x_1 + x_2 = v_0(2t - \tau) + \frac{1}{2} g(t^2 - 2t\tau + \tau^2 - t^2)$

or $x_{12} = 2v_0 t - v_0 \tau - g t \tau + \frac{1}{2} g \tau^2$

or $x_{12} = (2v_0 - g\tau)t - v_0 \tau + \frac{1}{2} g \tau^2$. *Ans.*

27. The time during which ball 1 falls is $t_1 = \sqrt{\dfrac{2h}{g}}$

$$= \sqrt{\frac{2 \times {\cdot}4\,4}{9{\cdot}8}} = 0{\cdot}3 \text{ s.}$$

The velocity equired by it on reaching the ground is

$$v_1 = gt = 9{\cdot}8 \times {\cdot}3 = 2{\cdot}94 \text{ m/s (downward).}$$

Immediately after collision the velocity becomes 2·94 m/s (upward).

The process is repeated as the collision is elastic. This is shown in the graph.

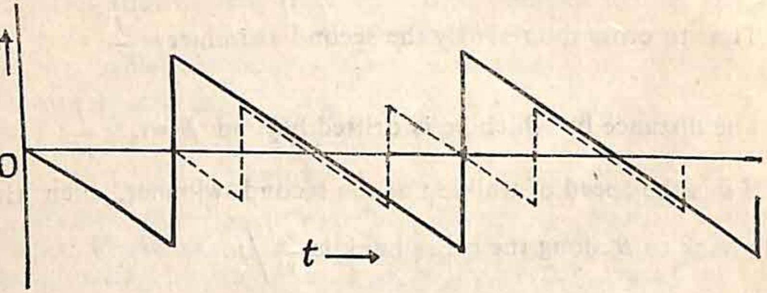


Fig. 1.26

The velocity of ball 2 on reaching the ground = v_2 (say).

Now $v_2=\sqrt{2gh_2}$

$$v_1=\sqrt{2gh_1}; \quad \therefore \quad \frac{v_2}{v_1}=\sqrt{\frac{h_2}{h_1}}=\sqrt{\frac{\cdot 11}{\cdot 44}}=\tfrac{1}{2}$$

or $$v_2=\frac{v_1}{2}.$$

If ball 2 is dropped exactly after ·3 seconds, the velocity of it will change exactly in the same way (shown by dotted curve) and the two will have identical velocities during the interval 0·3 s. *Ans.*

28. Let the lines intersect at O at the initial moment and then at O' after a short time t. Draw perpendiculars ON and OM on the displaced positions of the line. Then

$$ON=v_1t;\ OM=v_2t$$

and $$OO'=vt.$$

Obviously OO' is the diagonal of the parallelogram $OEO'F$.

$$OE=ON\operatorname{cosec}\alpha=v_1t\operatorname{cosec}\alpha$$

$$OF=OM\operatorname{cosec}\alpha=v_2t\operatorname{cosec}\alpha.$$

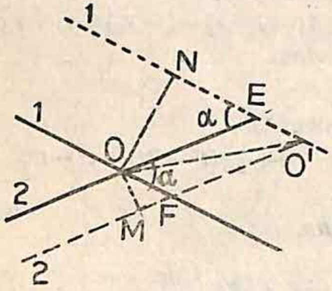


Fig. 1.27

$$\therefore OO'=\sqrt{OE^2+OF^2+2OEOF\cos\alpha}$$

or $v=(\sqrt{v_1^2+v_2^2+2v_1v_2\cos\alpha})\operatorname{cosec}\alpha.$ *Ans.*

29. The flag on a ship flies along the direction of the wind relative to the ship. Therefore, β is the angle made by the relative velocity of the wind relative to the ship.

Let us take direction of river flow as x-axis and the line perpendicular to it as y-axis.

Then $\vec{u} = u\,\vec{j}$

and $\vec{v}$ (velocity of ship) $= v\cos\alpha\;\vec{i} + v\sin\alpha\;\vec{j}$..

$$\therefore\quad \vec{u}_{wind-ship} = \vec{u} - \vec{v} = -v\cos\alpha\;\vec{i} + (u - v\sin\alpha)\,\vec{j}.$$

$$\therefore\quad \tan(\alpha+\beta) = \frac{u - v\sin\alpha}{-v\cos\alpha}$$

or $\quad -v\tan(\alpha+\beta)\cos\alpha = u - v\sin\alpha$

or $\quad v(\sin\alpha - \cos\alpha\tan\overline{\alpha+\beta}) = u,\quad$ or $\quad -v\sin\beta = u\cos(\alpha+\beta)$

or $\quad v = \dfrac{-u\cos(\alpha+\beta)}{\sin\beta} = \dfrac{u\sin(\alpha+\beta-\pi/2)}{\sin\beta}$. *Ans.*

30. Let $\angle CAB$ be α. Then $\cos\alpha = \dfrac{b}{\sqrt{a^2+b^2}}$.

To reach the point B the resultant velocity must be along AB. Take the river bank as x-axis and the line perp. to it as y-axis.

Then $\quad \vec{v}_{river} = v_0\,\vec{i}$.

Let the boat start at angle β with the river bank.

Then $\quad \vec{u}_{boat} = u\cos\beta\;\vec{i} + u\sin\beta\;\vec{j}$.

$$\therefore\quad \vec{v}\text{ (resultant velocity of boat)} = (v_0 + u\cos\beta)\,\vec{i} + u\sin\beta\,\vec{j}.$$

$$\therefore\quad \tan(\pi/2 - \alpha) = \frac{u\sin\beta}{v_0 + u\cos\beta},\quad\text{or}\quad \cot\alpha = \frac{u\sin\beta}{v_0 + u\cos\beta}$$

or $\quad v_0\cot\alpha + u\cos\beta\cot\alpha = u\sin\beta$.

Differentiating, we have

$$0 + \cot\alpha\left(\frac{du}{d\beta}\cos\beta - u\sin\beta\right) = \frac{du}{d\beta}\sin\beta + u\cos\beta.$$

When u is minimum, $\dfrac{du}{d\beta} = 0$.

$$\therefore\quad -u\cot\alpha\sin\beta = u\cos\beta$$

or $\quad \cot\alpha = -\cot\beta,\quad$ or $\quad \tan\alpha = -\tan\beta$

or $\quad \tan(\pi - \alpha) = \tan\beta,\quad$ or $\quad \alpha + \beta = \pi$.

If u_0 be the minimum value of u then

$$\cot\alpha = \frac{u_0\sin(\pi-\alpha)}{v_0 + u_0\cos(\pi-\alpha)}$$

or $\quad v_0\cos\alpha - u_0\cos^2\alpha = u_0\sin^2\alpha$

or $\quad u_0 = v_0\cos\alpha$

or $\quad u_0 = \dfrac{v_0 b}{\sqrt{a^2+b^2}}$. *Ans.*

31. Let the man start running at angle θ with the initial line AB joining him with the bus and meet the bus at D on the road.

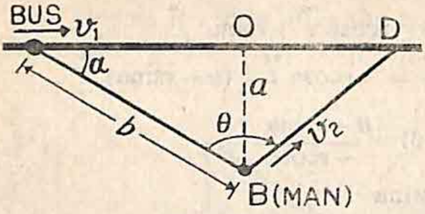


Fig. 1.28

Then $\angle ABD = \theta$

and $\angle DAB = \alpha$ (known)

$$= \sin^{-1}\frac{a}{b}.$$

Obviously,

$$AD = v_1 t_1;\ BD = v_2 t_2.$$

From 'sine property' of the triangle ADB,

$$\frac{AD}{\sin\theta} = \frac{BD}{\sin\alpha}.$$

$$\therefore\quad \frac{v_1 t_1}{\sin\theta} = \frac{v_2 t_2}{\sin\alpha},\quad \text{or}\quad \sin\theta = \frac{v_1 t_1}{v_2 t_2}.\sin\alpha = \frac{v_1 t_1}{v_2 t_2} \times \frac{a}{b}$$

$$\text{or}\quad \frac{t_1}{t_2} = \frac{v_2}{v_1}\cdot\frac{b}{a}.\sin\theta.$$

According to the condition, $t_1 \geqslant t_2$ or $\frac{t_1}{t_2} \geqslant 1$

or
$$\sin\theta \geqslant \frac{a v_1}{b v_2}.$$

But
$$\frac{a v_1}{b v_2} = \frac{60}{400} \times \frac{16}{4} = {\cdot}6.$$

$$\therefore\quad \sin\theta \geqslant {\cdot}6.$$

Hence θ may be any angle between 37° and 143°.

So the man should run either at 37° or 143° with the initial line joining him with the bus to reach simultaneously on the road or at any angle between 37° and 143° to reach earlier on the road.

32. See example 15. $t_{min} = 2\sqrt{\frac{(\alpha+\beta\ l}{2\alpha\beta}}$

$$v_{max} = \left(\frac{2\alpha\beta l}{\alpha+\beta}\right)^{\frac{1}{2}}$$

$$t_{min} = 2\sqrt{\frac{({\cdot}25+{\cdot}5)8\times 10^3}{2\times{\cdot}25\times{\cdot}5}} = 310\ \text{s} = 5\ \text{min. } 10\ \text{s.}\quad \textit{Ans.}$$

$$v_{max} = \sqrt{\frac{{\cdot}2\times{\cdot}25\times{\cdot}5\times 8\times 10^3}{{\cdot}75}} = 51{\cdot}6\ \text{m/s} = 186\ \text{km/h.}\quad \textit{Ans.}$$

33. See example 22. Here A and B are the two ships. The distance between ships reduces at the rate of $v_0(1-\cos\alpha)$. The speed of the projection of A on OX is $v_0\cos\alpha$. Hence the speed of B relative to C is $v_0 - v_0\cos\alpha = v_0(1-\cos\alpha)$. Thus BC increases at the rate of $v_0(1-\cos\alpha)$. Since the rate of increase of BC is the same as the rate of decrease of AB, the sum $(AB+BC)$ remains constant as the ships move. In the beginning $AB=a$ and $BC=0$ and at the end $AB=BC=$ distance between A and B (A now merges with C).

$$\therefore \quad a+0 = AB+BC$$

or

$$AB = \frac{a}{2}. \quad \textit{Ans.}$$

34. Let us fix up the frame of reference at the bank of the river with vertical upward as y-axis and the horizontal as x-axis.

Then $s = v\cos\theta t$ where θ is the angle of projection with the horizontal.

and

$$-h = v\sin\theta t + \tfrac{1}{2}(-g)t^2$$

or

$$-h = v\sin\theta \times \frac{s}{v\cos\theta} - \tfrac{1}{2}g.\frac{s^2}{v^2\cos^2\theta}$$

or

$$-h = s\tan\theta - \tfrac{1}{2}\ \frac{gs^2}{v^2}\ \mathrm{scc}^2\theta$$

$$= s\tan\theta - \tfrac{1}{2}\ \frac{gs^2}{v^2}(1+\tan^2\theta)$$

or

$$\frac{1}{2}\cdot\frac{gs^2}{v^2}\tan^2\theta - s\tan\theta - h + \tfrac{1}{2}\ \frac{gs^2}{v^2} = 0$$

or

$$\tan^2\theta - 2\,\frac{v^2}{gs}\tan\theta + \left(1 - \frac{2v^2h}{gs^2}\right) = 0$$

or

$$\tan\theta = \frac{v^2}{gs} \pm \sqrt{\frac{v^4}{g^2s^2} - \left(1 - \frac{2v^2h}{gs^2}\right)}.$$

This expression has a meaning if

$$\frac{v^4}{g^2s^2} \geqslant \left(1 - \frac{2v^2h}{gs^2}\right)$$

or

$$s^2 \leqslant \frac{v^4}{g^2} + \frac{2v^2}{g}$$

or

$$s \leqslant \frac{v}{g}\sqrt{v^2+2gh}$$

$$\therefore \quad s_{max} = \frac{v}{g}\sqrt{v^2+2gh}.$$

Putting this condition in the expression for $\tan\theta$ we have for maximum range

$$\tan\theta = \frac{v^2}{gs_{max}} = \frac{v}{\sqrt{v^2+2gh}}.$$

$$\theta = \tan^{-1}\frac{v}{\sqrt{v^2+2gh}}. \quad \textit{Ans.}$$

When $h=0$, $\theta=\tan^{-1}1=45°$. *Ans.*

35. $$\vec{v}_1 = v_0\cos\alpha_1\,\vec{i} + v_0\sin\alpha_1\,\vec{j}$$

and $$\vec{v}_2 = -v_0\cos\alpha_2\,\vec{i} + v_0\sin\alpha_2\,\vec{j}$$

$$\therefore \quad \vec{v}_{12} = v_0(\cos\alpha_1+\cos\alpha_2)\,\vec{i} + v_0(\sin\alpha_1-\sin\alpha_2)\,\vec{j}.$$

$$|\vec{v}_{12}| = v_0\sqrt{(\cos\alpha_1+\cos\alpha_2)^2+(\sin\alpha_1-\sin\alpha_2)^2}$$

$$= v_0\sqrt{2+2\cos(\alpha_1+\alpha_2)} = v_0\sqrt{4\cos^2\frac{\alpha_1+\alpha_2}{2}}$$

$$= 2v_0\cos\frac{\alpha_1+\alpha^2}{2}. \quad \textit{Ans.}$$

$$S_{12}\text{(distance between 1 and 2)} = |\vec{v}_{12}| \boxtimes \tau$$

or $$S_{12} = 2v_0\tau\cos\frac{\alpha_1+\alpha_2}{2}. \quad \textit{Ans.}$$

36. Let x, y be the co-ordinates of B. Then $\tan\theta = \frac{y}{x}$.

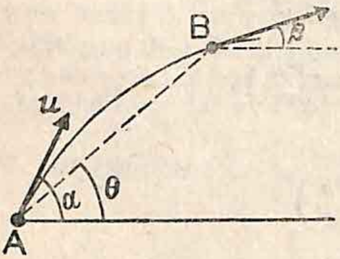


Fig. 1.29

$$x = u\cos\alpha t$$

$$y = u\sin\alpha t - \tfrac{1}{2}gt^2$$

u_x (x-component of velocity at B) $= u\cos\alpha$

u_y (y-component of velocity at B) $= u\sin\alpha - gt$.

$$\therefore \quad \tan\beta = \frac{u_y}{u_x} = \frac{u\sin\alpha - gt}{u\cos\alpha}$$

$$= \tan\alpha - \frac{gt}{u\cos\alpha} \quad \text{.. (i)}$$

and $$\tan\theta = \frac{u\sin\alpha t - \frac{1}{2}gt^2}{u\cos\alpha t} = \tan\alpha - \frac{1}{2}\,\frac{gt}{u\cos\alpha}$$

or $$2\tan\theta = 2\tan\alpha - \frac{gt}{u\cos\alpha}. \quad \text{.. (ii)}$$

Subtracting (i) from (ii),

$$2\tan\theta-\tan\beta=2\tan\alpha-\tan\alpha$$

or $$2\tan\theta=\tan\alpha+\tan\beta. \quad \textit{Proved.}$$

37. We have for horizontal and vertical motion of the ball,

$$s=u\cos\alpha t \quad \text{and} \quad h=u\sin\alpha . t-\tfrac{1}{2}gt^2.$$

Eliminating t we have

$$h=s\tan\alpha-\tfrac{1}{2}\,\frac{gs^2}{u^2\cos^2\alpha}, \quad \text{or} \quad u^2=\frac{gs^2}{2\cos^2\alpha(s\tan\alpha-h)}$$

or $$u^2=\frac{gs^2}{(s\sin2\alpha-h\cos2\alpha-h)}\,.$$

Obviously u is minimum when $(s\sin2\alpha-h\cos2\alpha-h)$ is maximum,

that is, $$\frac{d}{d\alpha}(s\sin2\alpha-h\cos2\alpha-h)=0$$

or $$2s\cos2\alpha+2h\sin2\alpha=0.$$

or $$\tan2\alpha=-\frac{s}{h}, \quad \text{or} \quad \tan(\pi-2\alpha)=\frac{s}{h}$$

or $$\alpha=\tfrac{1}{2}\left(\pi-\tan^{-1}\frac{s}{h}\right)$$

or $$\alpha=\frac{\pi}{2}-\tfrac{1}{2}\tan^{-1}\frac{s}{h}. \quad \textit{Ans.}$$

Since 2α is a second quadrant angle, $\sin2\alpha$ is positive and $\cos2\alpha$ is negative.

$$\therefore \quad \sin2\alpha=\frac{s}{\sqrt{h^2+s^2}} \quad \text{and} \quad \cos2\alpha=\frac{-h}{\sqrt{s^2+h^2}}$$

$$\therefore \quad u_{min}{}^2=\frac{gs^2}{\dfrac{s^2}{\sqrt{h^2+s^2}}+\dfrac{h^2}{\sqrt{h^2+s^2}}-h}=\frac{gs^2}{\sqrt{h^2+s^2}-h}$$

$$=g(h+\sqrt{h^2+s^2}).$$

$$\therefore \quad u_{min}=\sqrt{g(h+\sqrt{h^2+s^2})}. \quad \textit{Ans.}$$

38. The ball will be projected with velocity $u=\sqrt{2gh}$ at an angle $\left(\frac{\pi}{2}-\alpha\right)$ with the plane.

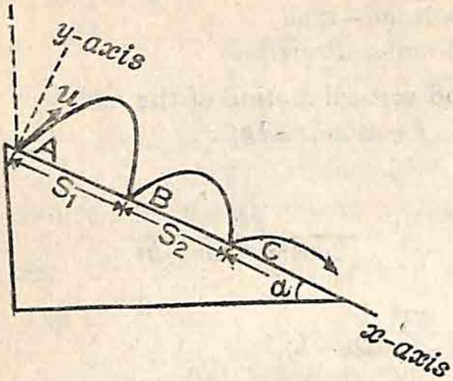


Fig. 1.30

Let us take *the plane as x-axis and perpendicular to it as y-axis.* Then acceleration along the plane is ($g\sin\alpha$) and that perpendicular to it is ($g\cos\alpha$).

$$\therefore \quad S_1 = u\sin\alpha t + \tfrac{1}{2}g\sin\alpha t^2$$

and $0 = u\cos\alpha t + \frac{1}{2}(-g\cos\alpha)t$ (as there is no displacement along y-axis in time t)

or $$t = \frac{2u}{g}.$$

and $$S_1 = \frac{2u^2\sin\alpha}{g} + \tfrac{1}{2}g\sin\alpha.\frac{4u^2}{g^2} = \frac{4u^2\sin\alpha}{g}$$

or $$S_1 = 8h\sin\alpha \quad (\because \quad u^2 = 2gh).$$

At the second impact at B,

velocity along the plane $= u\sin\alpha + g\sin\alpha t$

$$= 3u\sin\alpha \quad \left(\because \quad t = \frac{2u}{g}\right)$$

velocity perp. to the plane $= u\cos\alpha + (-g\cos\alpha)t$

$$= -u\cos\alpha.$$

After impact velocity along the plane remains unchanged but the normal velocity is reversed.

$\therefore$ Velocity along the plane after impact $= 3u\sin\alpha$
velocity perp. to the plane $= u\cos\alpha$.

$$\therefore \quad S_2 = (3u\sin\alpha)t + \tfrac{1}{2}g\sin\alpha t^2$$

and $$0 = u\cos\alpha t + \tfrac{1}{2}(-g\cos\alpha)t^2$$

or $$t = \frac{2u}{g}.$$

$$\therefore \quad S_2 = 3u\sin\alpha \times \frac{2u}{g} + \tfrac{1}{2}g\sin\alpha.\frac{4u^2}{g^2} = \frac{8u^2\sin\alpha}{g}$$

$$= 16h\sin\alpha \quad (\because \quad u^2 = 2gh).$$

Similarly, $S_3 = 24h\sin\alpha$, $S_4 = 32h\sin\alpha$.

$$\therefore \quad S_1 : S_2 : S_3 \ldots = 1 : 2 : 3 \ldots\ldots \quad \textit{Ans.}$$

39. Let the bomber dive at angle α with the horizontal at A. The bomb will be released at the same angle with the speed of the bomber.

So the bomb may be treated as a projectile projected at α with velocity v. In the figure $AB=h$,

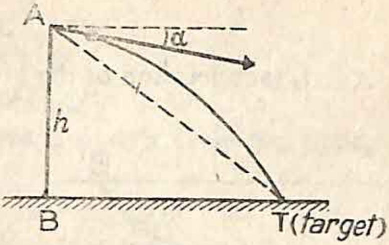


Fig. 1.31

$$BT=s=\sqrt{l^2-h^2}$$

$\therefore$ $s=v\cos\alpha t$

and $h=v\sin\alpha t+\frac{1}{2}gt^2$

or $h=s\tan\alpha+\frac{1}{2}\frac{gs^2}{v^2}(1+\tan^2\alpha)$

$$(\because \quad \sec^2\alpha=1+\tan^2\alpha)$$

or $$\tan^2\alpha+\frac{2v^2}{gs}\tan\alpha+\left(1-\frac{2v^2h}{gs^2}\right)$$

or $$\tan\alpha=-\frac{v^2}{gs}\pm\sqrt{\frac{v^4}{g^2s^2}-\left(1-\frac{2v^2h}{gs^2}\right)}\cdot$$

Since only positive solution has a meaning, we take

$$\tan\alpha=\sqrt{\frac{v^4}{g^2s^2}-\left(1-\frac{2v^2h}{gs^2}\right)}-\frac{v^2}{gs}.$$

Here $v=360\text{ kph}=100\text{ m/s}$

$$s=\sqrt{5-4}=1\text{ km}=1000\text{ m}.$$

$$\tan\alpha=\sqrt{\frac{10^8}{10^2\times10^6}-1+\frac{2\times10^4\times2\times13^3}{10\times10^6}}-\frac{10^4}{10\times10^3}$$

$$=2-1=1$$

or $$\alpha=45°. \quad \textit{Ans.}$$

40. Let v be the velocity of the river at a distance y from the bank. Then $v=ky$.

But $$v=v_0 \text{ when } y=\frac{c}{2}. \quad \text{(This is given)}$$

$$\therefore \quad v_0=k\frac{c}{2} \quad \text{or} \quad k=\frac{2v_0}{c}.$$

$$\therefore \quad v=\frac{2v_0}{c}y.$$

But $$y=ut.$$

$$\therefore \quad v=\frac{2v_0}{c}ut.$$

$\therefore$ Rate of change of velocity $=\frac{v}{t}=\frac{2v_0u}{c}$.

$\therefore$ a_x (acceleration of the boat along the water current) $=\frac{2v_0u}{c}$.

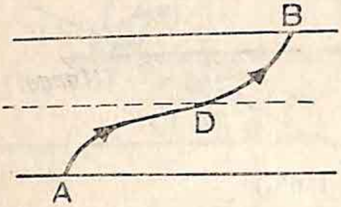


Fig. 1.32

$$\therefore \quad x=0\times t+\tfrac{1}{2}\left(\frac{2v_0u}{c}\right)t^2$$

and $\quad y=ut.$

$$\therefore \quad x=\frac{v_0u}{c}\cdot\frac{y^2}{u^2}=\frac{v_0}{uc}y^2$$

or $\quad y^2=\frac{uc}{v_0}x.$

This is equation of a parabola.

Thus the trajectory of the boat is a parabola *AD* up to the mid-stream. The other half of the trajectory (*DB*) is of the same nature as the first one.

When the boat is in the mid-stream, $y=\frac{c}{2}$.

$$\therefore \quad \frac{c^2}{4}=\frac{uc}{v}x$$

or $$x=\tfrac{1}{4}\,\frac{cv}{u}.$$

This gives the drift of the boat along the stream.

In the other half it is drifted by the same distance.

$\therefore$ The total distance by which the boat is drifted

$$=2x=\tfrac{1}{2}\,\frac{cv}{u}. \quad \textit{Ans.}$$

Solution and Hints to Test Questions

1. (i) (c). (ii) (d). (iii) (a). The flag will direct to the direction of the relative velocity of the wind relative to the steamer.

(iv) (c). (v) (d). (vi) (b). (vii) (c). (viii) (b).

(ix) (d). (x) (c).

2. (i) Angular momentum – ML^2T^{-1}
Torque – ML^2T^{-2}
Pressure – $ML^{-1}T^{-2}$
Conductivity – $MLT^{-3}K^{-1}$
Electromotive force – $ML^2T^{-2}Q^{-1}$,

(ii) (a) caesium. (b) Krypton ($_{36}K^{86}$), from $^2p_{10}$ to $5d_5$.
(c) area, normal to the area.

(iii) (a) false. (b) true. (c) true.

3. We have $t=\sqrt{x}+3$ or $x=t^2-6t+9$,

$$\therefore \quad v=\frac{dx}{dt}=2t-6.$$

Obviously velocity is zero when $t=3$ s.

$$\therefore \quad x=3^2-6\times3+9=0. \quad Ans.$$

4. We have $s=0\times t_1+\frac{1}{2}at_1^2$

and $v_0=at_1$

$$\therefore \quad s=\frac{1}{2}\cdot\frac{v_0}{t_1}\times t_1^2 \quad \text{or} \quad t_1=\frac{2s}{v_0}.$$

$$2s=v_0t_2 \qquad \text{or} \quad t_2=\frac{2s}{v_0}$$

$$3s=v_0t_3-\tfrac{1}{2}a't_3^2.$$

But $0=v_0-a't_3$ or $a'=\frac{v_0}{t_3}$.

$$\therefore \quad 3s=v_0t_3-\frac{1}{2}\cdot\frac{v_0}{t_3}.t_3^2=\tfrac{1}{2}v_0t_3 \quad \text{or} \quad t_3=\frac{6s}{v_0}.$$

Total distance $=s+2s+3s=6s$

Total time $=t_1+t_2+t_3=\frac{2s}{v_0}+\frac{2s}{v_0}+\frac{6s}{v_0}=\frac{10s}{v_0}$;

$$\therefore \quad \text{average speed}=\frac{6s}{\frac{10s}{v_0}}=\frac{3}{5}v_0. \quad Ans.$$

5. We have, $v^2-v_0^2=2g\times10$ (formula, $v^2-v_0^2=2as$)

or $v^2=v_0^2+20g$.

Let v' be the velocity after collision.

Then $\frac{1}{2}mv'^2=\frac{1}{2}\times\frac{1}{2}mv^2$

or $v'^2=\frac{1}{2}v^2$.

But $v'^2=2\times g\times 10=20g$ (This is given).

$\therefore\quad 20g=\frac{1}{2}(v_0^2+20g)$

or $v_0=\sqrt{20g}=\sqrt{20\times 9\cdot 8}=\sqrt{2\times 98}=2\times 7=14$ m/s. *Ans.*

(ii) It will rise to the same height because on returning to the point of projection the ball *would acquire the same downward velocity* and so the same thing would be repeated.

6. Let us take the direction of acceleration as x-axis and the line perpendicular to it as y-axis. Let the initial velocity V_0 make angle α with x-axis.

Then $x=V_0\cos\alpha t+\frac{1}{2}at^2$

and $y=V_0\sin\alpha t$ (there is no acceleration along y-axis)

$$\therefore\quad x=y\cot\alpha+\frac{1}{2}a\frac{y^2}{V_0^2\sin^2\alpha}.$$

This is equation of a parabola. *Proved.*

7. Let the rocket go up h with acceleration of 10 m/s².

Then $h=0+\frac{1}{2}\times 10\times 60^2=18000$ m.

The velocity acquired by it is $v=0+10\times 60=600$ m/s.

After complete burn-out of fuel the rocket motion is retarded by gravitation. Let it go up further by h'.

Then $600^2=2\times 9\cdot 8\times h'$

or $h'=18367$ m.

$\therefore$ Total height $=h+h'=18000+18367$

$=36367$ m

$=36\cdot 4$ km. *Ans.*

8. (a) We have $v=\alpha\sqrt{x}$.

Squaring, $v^2=\alpha^2 x$.

Differentiating, $2v\dfrac{dv}{dt}=\alpha^2.\dfrac{dx}{dt}$

$$=\alpha^2 v\qquad\left(\because\quad v=\frac{dx}{dt}\right)$$

or $$2\frac{dv}{dt}=\alpha^2$$

or $v=\frac{1}{2}\alpha^2 t+v_0$ (a constant).

When $t=0$, let $v=0$. Then $v_0=0$

and $v=\frac{1}{2}\alpha^2 t$. *Ans.*

a (acceleration) $=\dfrac{dv}{dt}=\frac{1}{2}\alpha^2$ (a constant). *Ans.*

(b) We have $u = \frac{dx}{dt} = \frac{1}{2}\alpha^2 t$, or $x = \frac{1}{4}\alpha^2 t^2$.

$$\therefore \quad S = \tfrac{1}{4}\alpha^2 t^2, \quad \text{or} \quad t = \frac{2\sqrt{S}}{\alpha}.$$

$$\therefore \quad <v> = \frac{S}{t} = \frac{\alpha\sqrt{S}}{2}. \quad \textit{Ans.}$$

9. (a) $x = a\sin\omega t$;

$$\therefore \quad v_x = \frac{dx}{dt} = a\omega\cos\omega t.$$

$y = a(1 - \cos\omega t)$;

$$\therefore \quad v_y = \frac{dy}{dt} = a\omega\sin\omega t.$$

$$\therefore \quad v = \sqrt{v_x^2 + v_y^2} = a\omega; \quad \therefore \quad S = v\tau = a\omega\tau. \quad \textit{Ans.}$$

(b) Let x be the instantaneous distance of the boat from the man and z be its distance from the shore. Then, if h is the height of the shore, we have $x^2 - z^2 = h^2$ (a constant).

Differentiating, $2x\frac{dx}{dt} = 2z\frac{dz}{dt}$.

$$\frac{dx}{dt} = v \text{ (given)}; \quad \frac{dz}{dt} = u \text{ (to be determined).}$$

$$\therefore \quad xv = zu, \quad \text{or} \quad u = \frac{x}{z}v.$$

But $\frac{z}{x} = \cos\alpha$. $\therefore$ $u = \frac{v}{\cos\alpha}$. *Ans.*

LESSON 2

Topics : **Laws of Motion**
Free-Body Diagrams
Momentum : Principle of Conservation of Momentum
Motion of Connected Bodies
Friction : Limiting and Static

2.1 Laws of Motion

The equation of motion of a particle

$$\vec{F} = m\vec{a} \quad \text{or} \quad \frac{d\vec{p}}{dt} \quad \text{or} \quad m\frac{d\vec{v}}{dt} \quad \text{or} \quad m\frac{d^2x}{dt^2} \quad \text{or} \quad m\ddot{x}$$

where $\vec{a}$ is the acceleration, $\vec{p}$ is the momentum and $\vec{v}$ is the instantaneous velocity of the particle and $\ddot{x}$ stands for double differentiation of x with respect to t.

$\vec{p}$ (momentum) $= m\vec{v}$. This is a vector quantity.

Note. If more than one force act on the particle, then $\vec{F}$ is the resultant of all these forces.

If no force acts on the particle, its motion will remain unchanged. Remember that force is not necessary to maintain the motion; force is necessary to change the motion. This aspect is described by the word 'inertia'.

For a system of particles or rigid body the equation of motion is

$$\vec{F}_{external} = m\vec{a}_{cm} \quad \text{or} \quad m\frac{d\vec{v}_{cm}}{dt}$$

where 'cm' stands for 'centre of mass'.

A special case arises when we deal with a system of varying mass, such as a rocket. The equation of motion is

$$F_{external} = m\frac{dv}{dt} + u_{rel}\frac{dm}{dt}$$

where u_{rel} is the relative velocity of the leaving mass relative to the main mass. The force $u_{rel}\frac{dm}{dt}$ is the 'thrust' generated by the leaving mass.

Thrust generated by the leaving mass $= u_{rel}\dfrac{dm}{dt}$.

2.2 Free-body Diagram : Force and Environment

In solving problems in dynamics confusion arises in considering motion of a body when it is acted upon by a number of forces. To overcome this confusion a scheme is available called 'free-body' diagram. In this scheme a separate diagram of the body is drawn and in this diagram different forces exerted by the bodies in the environment are shown. The number of forces must be the same as the number of bodies in the environment. Bodies in contact with the given body exert 'contact force'. Bodies which are not in immediate contact may also exert force on the body and that force is the 'field force'. The weight of a body is an example of 'field force'. There may be other field forces such as 'electric force', 'magnetic force' etc. To solve prolems by this scheme the following steps are recommended.

(i) Make a separate diagram of the body, showing all the forces—contact forces as well as the field forces.

(ii) See carefully that you have considered all the forces. Remember that the number of forces must be the same as the number of bodies in the environment.

(iii) Now project (resolve) all the forces along and perpendicular to the direction in which motion takes place and apply 'force = mass × acceleration' in each direction.

D' Alembert's Principle. This principle is a very useful formalism which considerably simplifies many difficult problems in dynamics. By this principle a dynamic system can be reduced to a static system by introducing a force called 'inertia force', equal to the product of mass of the system and its acceleration at the centre of mass of the system and directed opposite to the acceleration of the system. This force will be in addition to the real forces, that is, the forces accountable to its environments. Forces exerted by real bodies in the environment are called real forces.

2.3 Momentum and Conservation of Momentum

$$\vec{p}\ (\text{momentum of a particle}) = m\,\vec{v}.$$

When no external forces act on a system of particles $\vec{p} =$ a constant.

This is known as the principle of conservation of momentum.

Since momentum is a vector quantity, the above principle may be applied separately for two mutually perpendicular directions at the same point. This becomes very useful in problems involving collisions.

2.4 Motion of Connected Bodies : Tension of a String

When several bodies are connected by unstretchable strings passing over several pulleys, motion of the bodies takes place under 'constraints'. The principal constraint is that lengths of the strings do not change. This constraint determines whether different parts will have the same acceleration or different accelerations. The strings will be under tension. The tension of a string is the force exerted by an element of it over the adjoining element across the boundary of the elements. If the string is light (weightless) its tension throughout its length is the same; otherwise it will change from point to point. The tension of a string on the two sides of a pulley is the same if the pulley is frictionless; otherwise the tensions on the two sides of the pulley will be different. Make free-body diagrams of the bodies and solve for the sought quantity.

At any point on a string the tension may be considered to be acting in either direction.

2.5 Friction

Force of friction arises between two surfaces in contact, when some reaction exists between them. By nature, the force of friction opposes any motion or tendency of motion between the surfaces. Let N be the normal reaction between two surfaces in contact; let f = force applied to one of the bodies (the other being rigidly fixed); let F = force of friction. If f is gradually increased from zero, F will also increase, so that f and F are always equal and opposite. No motion occurs. However, F has a certain maximum value, called the 'limiting friction', given by

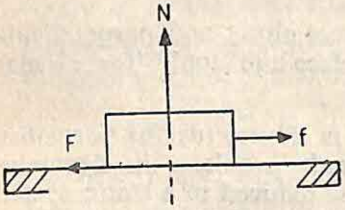


Fig. 2.1

$$F_{lim} = \mu_s . N,$$

μ_s = coefficient of static friction. Its value depends only on the nature of the surfaces in contact, being larger for rough surfaces, and smaller for smooth surfaces. When the bodies in contact are in motion, $F_k = \mu_k N$ where μ_k is the coefficient of kinetic friction. Remember μ_k is always less than μ_s. Thus, frictional force is always less than limiting frictional force and is equal to the applied force, provided it (applied force) is less than the limiting frictional force.

If, however, it exceeds the limiting frictional force, the frictional force is the limiting frictional force itself.

Forces of reaction are commonly encountered in most problems. Reaction is a contact force, and will arise whenever two surfaces are in contact. Let us first consider 'normal reaction'. When two surfaces are in contact, the normal reaction will act along the 'common normal to the two surfaces'. Figs. 2.2 to 2.5 show some applications of this principle.

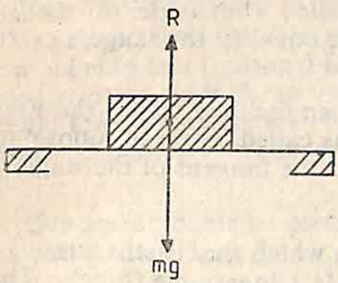


Fig. 2.2

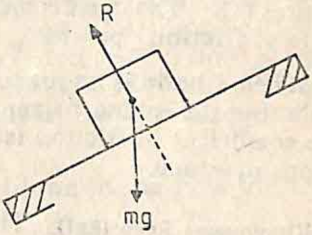


Fig. 2.3

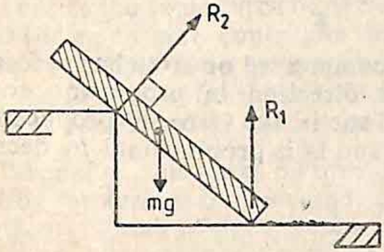


Fig. 2.4

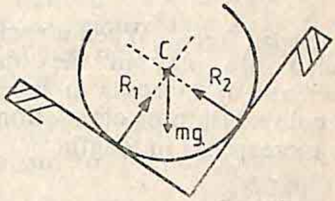


Fig. 2.5

This rule gives us only the direction of the normal reaction; its magnitude has to be obtained from the equations of motion.

If two surfaces in contact are 'fixed' or 'hinged', the reaction in general will not be normal. To find the reaction in such cases, use the principle that if a system is in equilibrium under the influence of three forces, the lines of action of the forces must pass through the same point. An example of this is given in Fig. 2.6. The word 'reaction of a surface' sometines denotes the resultant of the normal reaction of a surface and the force of friction at the surface. In Fig. 2.7, N = normal reaction, F = force of friction due to spinning wheel, R = reaction of road = resultant of N and F.

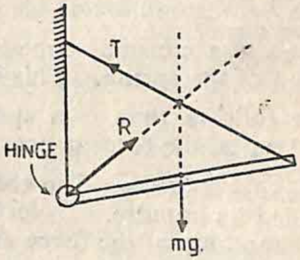


Fig. 2.6

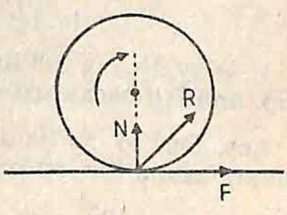


Fig. 2.7

Angle of Friction (λ) and Angle of Repose (θ). When a body is resting on an inclined plane the angle made by the reaction of the plane on a body when it is about to slide is called the angle of limiting friction (λ). The coefficient of friction is equal to the tangent of the angle of friction $\mu_s = \tan\lambda$.

When a body is about to slide down an inclined plane, the angle made by the plane to the horizontal is called 'angle of repose' (θ). The coefficient of friction is also equal to the tangent of the angle of repose $\mu_s = \tan\theta$.

Kilogramme force (kgf). The force with which the earth attracts a 1 kg body towards its centre is called 'a kilogramme force'. Thus

$$1 \text{ kg } f = \frac{\text{Force in newton}}{g}.$$

Spring Force. When a spring is compressed or stretched a force is called into play in the opposite direction in proportion to the decrease or increase in length. That is, the force 'F' produced due to compression or elongation of a spring is proportional to decrease or increase 'x' in length.

That is, $$F = -kx$$

where k is a constant called 'force constant' of the spring. It depends on the length 'l', material of the spring and area of cross-section. If l is the length of the spring, then

$$F = -\frac{\lambda}{l}x$$

where λ is a constant depending only on material and transverse section of the spring. This is called 'modulus' of the spring.

It follows that if a spring of force constant k is cut into two equal parts, the force constant of each part is $2k$.

Impulse of a Force. The change in momentum produced by a force is called its impulse. In terms of force it is equal to the product of the magnitude of the force and the duration of action of the force.

$$\text{Impulse} = m(v_f - v_i) = F.\Delta t.$$

CRITICAL QUESTIONS

1. Why do you fall forward when a moving train decelerates to a stop and fall backward when a train accelerates from rest ?

Ans. Due to inertia of motion of the upper part of the body. Due to inertia for rest of the lower part of the body.

2. When walking on ice, is it better to take short or long steps ? Explain. **(NDA 1977)**

Ans. Ice is almost frictionless. In walking there must be sufficient friction to prevent slipping of feet. Frictional force depends on the normal reaction and normal reaction in its turn depends on the inclination of the reaction of the ground with the vertical. If we take long steps the inclination of the reaction to the vertical becomes large and hence less frictional force. If we take short steps, then reaction remains close to the vertical. This helps to develop more frictional force on a slippery road such as ice and makes walking possible. This is why while walking on ice it is better to take short steps.

3. Can sailing boats be propelled by air blown at the sails from a fan attached to the boat ?

Ans. No. Consider (boat+fan+air) as a system. This system is free from external force. The force exerted by fan on the sail through air, sail exerts the same force on the air and these are internal forces. Internal forces do not contribute to the motion of a system of particles or bodies. For a system we have $F_{ext}=(\Sigma m)a$. Since $F_{ext}=0$, $a=0$ or $v=$ constant. So if the boat is initially at rest, the air blown from the fan cannot start motion in the system.

4. When a person walks on a rough surface, the frictional force exerted by the surface on the person is opposite to the direction of his motion. True or false ? (IIT 1981)

Ans. False. When a man walks, he rubs his foot or footwear in the backward direction. The rough surface exerts an equal and opposite force. Thus frictional force on the man is in the direction of the man.

5. A rocket moves forward by pushing the surrounding air backwards. True or false ? (IIT 1980)

Ans. False. A rocket moves by the thrust generated by the gas ejected at the rear end of it.

6. There is a limit beyond which further polishing of a surface increases rather decreases frictional resistance. Can you explain this ?

Ans. If there is over polishing, molecules of the surfaces come within molecular range and so strong molecular force of attraction arise between them.

7. Is the tension of a string the same throughout its length if it is not massless ?

Ans. No. Tension varies from point to point if the string is not massless.

8. Is the tension of a string on either side of a pulley the same ? If so, under what conditions ?

Ans. Yes, the tension is the some on either side of the pulley if the pulley is frictionless.

9. Is it unreasonable to expect a coefficient of friction to exceed unity ?

Ans. No, because the coefficient of friction is given by $u=\tan\lambda$ where λ is the angle of friction. Since tan of an angle may be anything from zero to infinity, it is not unreasonable to expect μ to exceed unity.

10. Explain how the range of your car's headlights limits the safe driving speed at night.

Ans. When a car moves, there is rolling friction between the tyres and road. Rolling friction is small in comparison to sliding friction. When brake is applied, the wheels are clamped and so there arises enormous sliding friction. This sliding friction stops the car. If l is the range of the headlight, the car must be stopped within this distance. Let M be the mass of the car and μ_s is the coefficient of sliding friction, then limiting frictional force $=\mu_s Mg=Ma$ where a is the retardation produced.

or
$$a=\mu_s g.$$

But
$$v^2_{safe\ speed}=2al=2\mu_s gl.$$

$$\therefore\quad v_{safe\ speed}=\sqrt{2\mu_s gl}.$$

11. When a lift moves up with uniform acceleration the weight of a man apparently increases. Why ? **(NDA 1979)**

Ans. The apparent weight indicated by a spring balance is the force with which it is stretched by the man. Let R be the force with which the man stretches or compresses the spring. Then the spring exerts the same force on the man. Considering 'free-body' diagram of the man we have

$$R-mg=ma \text{ or } R=mg+ma.$$

This is why the weight apparently increases.

12. Why is it easier to pull than to push a lawn roller ?

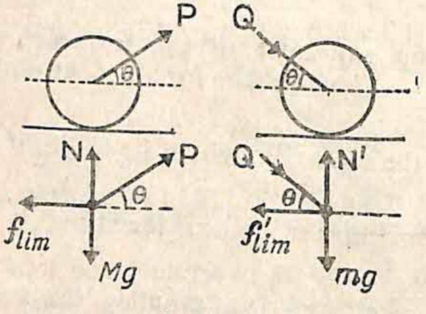


Fig 2.8

Ans. Push is a force directed inward (towards the centre) and 'pull' is a force directed away from the centre. Let P be the force required to move the roller by pulling and Q be the force required to move it by pushing applied at the same inclination θ to the horizontal. Considering 'free-body diagrams' in the two cases we have

$$N+P\sin\theta=mg \qquad\qquad N'-Q\sin\theta=mg$$

$$\text{and}\quad P\cos\theta=f_{lim}=\mu_s N \qquad\qquad \text{and}\quad Q\cos\theta=f'_{lim}=\mu_s N'$$

$$\text{Hence } P=\frac{\mu_s mg}{\cos\theta+\mu_s\sin\theta}. \qquad\qquad \text{Hence } Q=\frac{\mu_s mg}{\cos\theta-\mu_s\sin\theta}.$$

$$\therefore\quad P<Q.$$

This is why it is easier to pull than to push a lawn roller.

13. A goods train has 25 compartments. Will the tension in coupling between the fourth and the fifth compartment be the same as that in the coupling between the twenty-first and the twenty-second compartment ?

Ans. No. Let P be the pull of the engine. Considering 'free-body' diagram of the first four compartments we have

$$P-4f-T=4ma$$

where f=frictional force on each compartment, m=mass of each compartment, a=acceleration of the train, T=tension. Considering 'free-body' diagram of the 21 compartments we have

$$P-21f-T'=21ma.$$

$$\text{Hence}\qquad T=P-4(f+ma) \text{ and } T'=P-21(f+ma).$$

$$\therefore\quad T<T'.$$

14. A student standing on the large platform of a spring scale notes his weight. He then takes a step on this platform and notices that the scale reads less than his weight at the beginning of the step and more than his weight at the end of the step. Explain.

Ans. When the student takes a step on the platform of the balance there is sudden change in motion (downward) in the beginning and a sudden upward change at the end of the step So if R is force exerted by the spring of the balance on the student, then in the beginning of the step $(mg-R)>0$ or $R<mg$. So the scale reads less. At the end of the step, $R-mg>0$ or $R>mg$. So the scale reads more.

15. Why is it difficult to walk on sand ?

Ans. Because of yielding nature of sand surface, it cannot exert as much forward thrust as a hard ground.

16. Why a clean hole is made when a bullet is fired at a glass window pane but it is broken into pieces by a stone ?

Ans. The speed of the bullet is very high. Due to large inertia for motion a clean hole is made. When a stone is thrown due to low inertia for motion of the stone, it cannot cut through it.

17. The pull of a horse on a wagon is equal and opposite to the pull of the wagon on the horse. If the horse can never exert a greater force on the wagon than it exerts on the horse, how is the wagon set moving ? Explain.

Ans. The motion of a system of bodies depend on the external forces acting on the system and NOT on the internal action and reactions of the different parts of the system. Consider wagon+horse as a system. Make freebody diagrams of wagon and horse. The earth is an external body for the system. The gravitational pull due to the earth and the frictional force are external to the system. See below carefully how motion of the system does not depend on the internal action and reaction of the system.

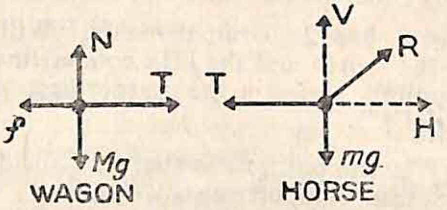


Fig. 2.9

Equation of motion of the wagon	Equation of motion of the horse	System = wagon + horse
$T-f=Ma$	$H-T=ma$	$H-f=(M+m)a$
$N=Mg$	$V=mg$	$N+V=(M+m)g.$

Thus it is seen that the motion of the system follows when the horizontal component of reaction of the earth exceeds the limiting frictional force on the wagon and never depends on what amount of force the horse applies on the wagon and the wagon on the horse.

18. Can we predict the direction of motion from the direction of force ?

Ans. No. According to Newton's 2nd law, the resultant force acting on a body determines the magnitude and direction of its acceleration and not its velocity. So from the direction of force we cannot predict direction of motion. For example, in circular motion, motion takes place at right angles to the force but acceleration is definitely along the direction of the force, that is, towards the centre of the circle.

19. Why does a fielder lower his hand while taking a 'catch' ?

Ans. The ball lands on the hand of the fielder with tremendous speed. To stop it the fielders must apply the same impulse with which the batsman sends it to him. By lowering his hand he manages to lengthen the duration of action of the force exerted by him on the ball. Thus the same impulse can be applied by the fielder by a comparatively much smaller force. To secure this advantage a fielder lowers his hands while taking a 'catch'.

20. Arrange μ_s-coefficient of static friction, μ_k-coefficient of kinetic friction and μ_r-coefficient of rolling friction in ascending order.

Ans. $\mu_r < \mu_k < \mu_s$.

21. Can the angle θ in the figure be ever exactly 180° ? If not, why ?

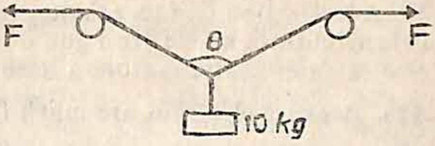


Fig. 2.10

Ans. No. For equilibrium,

$$2F\cos\frac{\theta}{2}=10g, \quad \text{or} \quad \theta=2\cos^{-1}\left(\frac{5g}{F}\right).$$

$$\therefore \quad \theta \neq 180°.$$

22. Two men, each of mass m, stand on the edge of a stationary car and jump off with a horizontal velocity u relative to the car first simultaneously and then one after the other. If friction be negligible, in which case will they impart greater velocity to the car ?

Ans. Let M be the mass of the car excluding the mass of the men. Let u_1 be the absolute velocity of the man to the right after one man jumps off the car and u_2 be the absolute velocity of the car to the left. Then $u=u_1-(-u_2)=u_1+u_2$.

Considering conservation of horizontal momentum of the system (man+car) to the left (this is justified because the system is free from external forces along the horizontal)

$$(M+2m)0=(M+m)u_2+m(-u_1), \quad \text{or} \quad u_2=\frac{mu}{M+2m}.$$

Considering conservation of momentum when the second man jumps off

$$(M+m)u_2=Mu_2'+m(-u_1')$$

and $$u=u_1'-(-u_2')=u_1'+u_2'$$

or $$(M+m)\times\frac{mu}{M+2m}=Mu_2'-m(u-u_2')$$

or $$u_2'=mu\left(\frac{1}{M+2m}+\frac{1}{M+m}\right). \quad \ldots (1)$$

When both men jump off simultaneously

$$(M+2m)\times 0=Mu_2+2m(-u_1)$$

and $$u=u_1-(-u_2)=u_1+u_2.$$

Hence $$u_2=\frac{2mu}{M+2m}=mu\left(\frac{1}{M+2m}+\frac{1}{M+2m}\right). \quad \ldots (2)$$

Comparing (1) and (2), $u_2'>u_2$.

So *they can import greater velocity by jumping one after the other.*

23. Acceleration due to gravity is independent of mass. Why then electron from an electron gun does not fall as much because of gravity as water particles from a hose ?

Ans. Because electrons are much faster than water particles from a hose.

24. A bird is in a wire cage hanging from a spring balance. Is the reading of the balance when the bird is flying about greater than, less than, or the same as that when the bird sits in the cage ?

Ans. The balance reads the same in both the cases because for the system (cage+bird+air) action and reaction of the internal parts neutralise and so the external forces on the system remain unchanged whether or not the internal parts act and react.

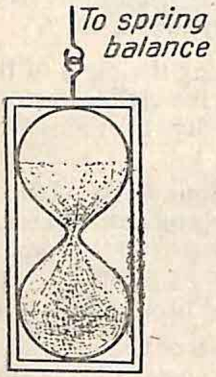


Fig. 2.11

25. A sand glass is being weighed on a sensitive balance, first when the sand is dropping in a steady stream from the upper to the lower part and then again after the upper part is empty. Are the two weights the same or not ? Explain your answer.

Ans. The two weights are the same. The stream of sand leaving the upper part exerts force upward and the stream of sand reaching the lower part exerts the same downward force, that is, the sand stream being continuous from upper to lower part of vessel, the two forces become the internal forces of the system and so equilibrium is not effected. However, at the moment the last element of the stream leaves the upper part the falling elements exert greater force than their weights on account of their gained momentum under gravity and so momentarily the balance will read more. But when all the sand has collected and there is no stream, the balance reading will be steady and again it will give the same reading.

26. How can you reconcile the sailing of a sailboat into the wind with the conservation of momentum principle ?

Ans. Considering boat+wind as a system free from external force, we can reconcile the sailing of a sailboat into the wind. Here momentum of the wind is transferred to the boat.

27. Both the following statements are correct; explain them. Two

teams having a tug-of-war must always pull equally hard on one another. The team that pushes harder against the ground wins.

Ans. The harder each team pulls the other, the greater the force it exerts on the other team through the rope and so to win each team must pull hard on the other team. The harder the team pushes against the ground, the greater the reaction of the ground and the greater the component of the reaction helping the team to pull over the other team to their side.

28. A massless rope is passed over a frictionless pulley. A monkey holds on to it and a mirror, having the same weight as the monkey, is attached to the other end of the rope at the monkey's level. Can the monkey ever get away from his image? Explain.

Ans. No. Since the two weights are equal, if the monkey let loose the rope, both will have the same downward acceleration. If the monkey goes up with acceleration, tension in the rope must be greater than the weight of the monkey and the same tension is effective on the mirror. The mirror being of the same mass will have the same upward acceleration. Thus the monkey can never get away from his image.

29. Why does a gun appear to have a greater 'kick' when fired with the butt held loosely against the shoulder than when held tightly?

Ans. When held loosely, the gun alone recoils with velocity $\frac{mv}{M}$ where m is the mass of the bullet and v is its velocity and M is the mass of the gun. But when held tightly, the gun and the man form a single rigid body and so the velocity of recoil is $\frac{mv}{M+W}$ where W is the mass of the man. The velocity in the former case being large, the man experiences a greater kick.

30. A uniform rod AB is placed horizontally on forefingers of left and right hand as shown in the figure. The left finger is at M and the right finger is at the extreme right end B. If you want to move your fingers inwards towards the centre O of the rod you will see that first your right finger moves inwards and then both the fingers simultaneously when the two are equidistant from the centre O. Explain why is it so.

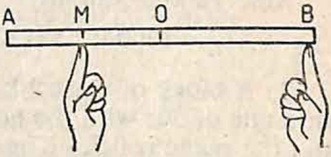


Fig. 2.12

Ans. The normal reaction on the left finger is greater than that on the right finger because of its closeness to the centre of gravity. Hence frictional force on the left finger is greater than that on the right finger and so the right finger moves in first. When they are equidistant, the frictional force is the same on both the fingers. This is why they move in simultaneously.

31. Several identical elastic balls are suspended in a row as shown in the figure. If one or two or three etc. balls are moved aside and then released, they collide with the remaining balls with a certain common velocity v. It is found that one or two or three etc. balls respectively from the extreme right move out to the right with the same velocity, though to conserve momentum it is sufficient for a single ball to move out with v, $2v$, $3v$ etc. respectively when one or two or three balls hit from the left. Explain why ?

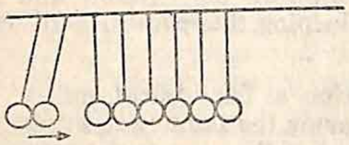


Fig. 2.13

Ans. The same number of balls move out to conserve energy and momentum. If two balls hit with velocity v and only one ball moves out with $2v$, momentum is conserved no doubt but energy is not conserved. On the other hand if two balls move out, both momentum and energy are conserved.

EXAMPLES

OBJECTIVE QUESTIONS

1. A jet engine works on the principle of (a) conservation of mass, (b) conservation of energy, (c) conservation of linear momentum, (d) conservation of angular momentum. **(I.I.T. 1975)**

Ans. (c).

2. A car of mass 1200 kg is brought to rest from a speed of 72 kph within 80 m. The retarding force on the car is (a) 2000 N, (b) 3000 N, (c) 300 N, (d) 200 N.

Ans. 72 kph $=20$ ms^{-1}. $20^2=2\,a.\,80$ or $a=2{\cdot}5$ ms^{-2}.

$F=1200\times2{\cdot}5=3000$ N. So the answer is (b).

3. A block of mass 2 kg rests on a rough inclined plane making an angle of 30° with the horizontal. The coefficient of static friction and the plane is 0·7. The frictional force on the block is (a) 9·8 N, (b) $0{\cdot}7\times9{\cdot}8\times\sqrt{3}$ N, (c) $9{\cdot}8\sqrt{3}$ N, (d) $0{\cdot}7\times9{\cdot}8$ N. **(I.I.T. 1980)**

Ans. Make a free-body diagram. Let N be the normal reaction.

$N=2g\cos30^\circ=2g\sqrt{3}/2=\sqrt{3}g$. $\therefore$ $f_{lim}=0{\cdot}7\times\sqrt{3}g$.

Force down the plane $=2g\sin30^\circ=g$. Since the force down the plane is less than the limiting frictional force, frictional force is equal to the force down the plane. So the answer is (a).

4. Figure below shows a spring balance with additional provision to clamp one end tightly by the screw S. Initially the pointer reads zero and the screw is tightened to clamp the spring at one end. If now two 10 kg weights are put on the pans, does the scale read (a) 0 kg, (b) 5 kg, (c) 10 kg, (d) 20 kg ?

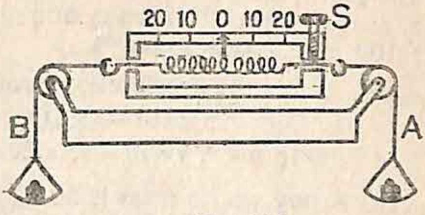


Fig. 2.14

Ans. The spring is stretched by 10 kgf force and so the answer is (c).

5. If in the figure of problem 4 the screw is unclamped and then 10 kg weights are put on the pans, does the scale read (a) 0, (b) 5 kg, (c) 10 kg, (d) 20 kg ?

Ans. Now the spring stretches equally on either side. So the pointer does not shift. Therefore, the answer is (a).

6. A spring balance is clamped to the ceiling and a 40 kg weight is hung from its hook. The scale reads 40 kg. The ceiling exerts (a) a force greater than 40 kg, (b) zero force, (c) a force less than 40 kg, (d) a force exactly equal to 40 kg.

Ans. Considering free-body diagram of the spring we find that the ceiling exerts exactly the same force. So the answer is (d).

7. Two weights are suspended from a string thrown over a light frictionless pulley. The mass of one weight is ·200 kg. If a heavy weight is attached to its other end, the tension in the string is (a) zero, (b) ·200 kgf, (c) ·400 kgf, (d) ·600 kgf.

Ans. $T=\dfrac{2\times \cdot 200\,Mg}{\cdot 200+M}$ where M is the mass, suspended at the other end. Since the weight is heavy M is very large.

$$\therefore \quad T=\frac{\cdot 400g}{1+\dfrac{\cdot 200}{M}}=\cdot 400g \text{ (when } M\to\infty).$$

So the answer is (c).

8. A large block of wood is dragged along the platform of a railway station with 30 kgf force. The coefficient of friction between wood and concrete of platform is 0·5. The force of friction is (a) 30 kgf, (b) 20 kgf, (c) 10 kgf, (d) zero.

Ans. $f_{lim}=0{\cdot}5\times 40g=20g\ \text{N}=20$ kgf. Since the applied force exceeds the limiting frictional force, the effective friction is the limiting friction itself. So the answer is (b).

9. While launching a rocket of mass 2×10^4 kg from the surface of the earth, a force of 5×10^5 N is applied for 20 s. The velocity of the rocket after 20 s is (a) 500 ms^{-1}, (b) 3000 ms^{-1}, (c) 200 ms^{-1}, (d) 100 ms^{-1}. ($g=10\ \text{ms}^{-2}$).

Ans. If a is the acceleration produced,

$$5\times10^5-2\times10^4\times10=2\times10^4\times a$$

or $a=15\ \text{ms}^{-2}$, $v=at=15\times20=300\ \text{ms}^{-1}$. So the answer is (b).

10. A boy whose mass is 50 kg stands on spring weighing machine inside a lift. The lift starts to ascend with an acceleration of 2 ms^{-2}. The reading of the machine ($g=10\ \text{ms}^{-2}$) is (a) 50 kg, (b) zero, (c) 49·5 kg, (d) 60 kg.

Ans. Here $R-50\times10=50\times2$, or $R=600\ \text{N}=60$ kgf. So the answer is (d).

11. An elevator weighs 4000 kg. When the upward tension of the supporting cable is 48000 N, the acceleration of the lift ($g=10\ \text{ms}^{-2}$) is (a) 5 ms^{-2}, (b) 3 ms^{-2}, (c) 2 ms^{-2}, (d) 1 ms^{-2}.

Ans. Here $48000-4000\times10=4000\times a$, or $a=2\ \text{ms}^{-2}$. So the answer is (c).

11. A heavy body of mass 25 kg is to be dragged along a horizontal plane $\left(\mu=\dfrac{1}{\sqrt{3}}\right)$. The least force required is (a) 25 kgf, (b) 2·5 kgf, (c) 12·5 kgf, (d) 50 kgf.

Ans. Let us first deduce a general formula. Let M be the mass of the load and P is the applied force. Let θ be the angle at which P is applied to the vertical.

Since there is no motion along the vertical we have

$$N+P\cos\theta=Mg.$$

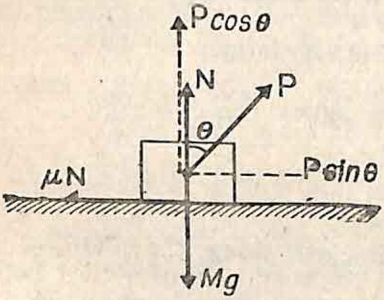


Fig. 2.15

The body will move when horizontal forward force is greater or equal to the limiting frictional force, that is,

$$P\sin\theta\geqslant\mu N$$

or $$P\geqslant\frac{\mu Mg}{\sin\theta+\mu\cos\theta}.$$

Put $\mu=\tan\lambda$, where λ is a constant called angle of friction.

$$\therefore\quad P\geqslant\frac{Mg\sin\lambda}{\sin(\theta+\lambda)}.$$

$$\therefore\qquad P_{min}=\frac{Mg\sin\lambda}{\sin(\theta+\lambda)}.$$

Further P_{min} depends on θ. The value of P_{min} is minimum when $\sin(\theta+\lambda)$ is maximum. But the maximum value of $\sin(\theta+\lambda)$ is 1.

$\therefore$ The least force needed $= Mg\sin\lambda = Mg\sin 30°$

$= 12{\cdot}5$ kgf. *Ans.*

$(\because\ \mu=\tan\lambda,\ \therefore\ \tan\lambda=\frac{1}{\sqrt{3}};\lambda=30°)$

So the answer is (c).

12. A block of mass m is supported by a cord C from the ceiling, and another cord D is attached to the bottom of the block (see figure). If D is pulled on steadily (a) C breaks earlier than D (b) D breaks earlier than C, (c) both break simultaneously, (d) can never be said which one will break earlier.

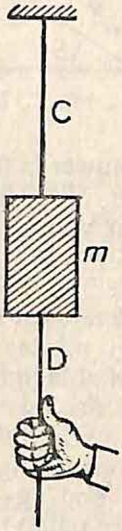


Fig. 2.16

Ans. Consider a (free-body) diagram of the block. Since it is pulled steadily, no acceleration is produced in m.

$\therefore\ T_D+mg-T_C=0$

or $T_C-T_D=mg$

$\therefore\ T_C>T_D$.

Thus C will break earlier than D because its tension will reach 'breaking tension' of the cord earlier than that of D. So the answer is (a).

13. The coefficient of friction between a block and a plane is $\frac{1}{\sqrt{3}}$. If the plane is gradually inclined, the block will begin to slide when the inclination of the plane is (a) 60°, (b) 45°, (c) 15°, (d) 30°.

Ans. Since coefficient of friction is $\frac{1}{\sqrt{3}}$, $\tan\lambda=\frac{1}{\sqrt{3}}$ or $\lambda=30°$.

But M is also given by $\mu=\tan\theta$; $\therefore\ \theta=30°$.

So the answer is (d).

14. A block is about to slide down an inclined plane when its inclination to the horizontal is θ. If now a 5 kg weight is put on the block, (a) it is still about to slide down the plane, (b) it is not to slide down the plane unless the inclination is increased, (c) it is not to slide down the plane unless the inclination is decreased, (d) it will never slide down whatever be the inclination.

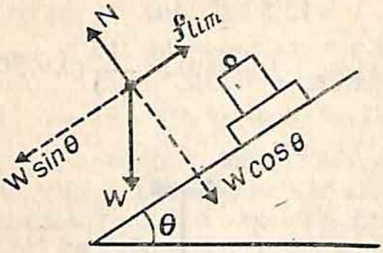


Fig. 2.17

Ans. Suppose W is the total weight (weight of block +weight on block). Make a 'free-body' diagram of W. Projecting forces along and perpendicular to the plane, we have

$N = W\cos\theta$ and $f_{lim} = W\sin\theta$.

But $f_{lim} = \mu N$.

$$\therefore \mu = \frac{W\sin\theta}{W\cos\theta} = \tan\theta.$$

Thus O is independent of W.

So the anwer is (a).

15. A body of mass 1 kg is suspended by a thread. If the body is taken up with acceleration 5 ms^{-2}, the tension of the thread ($g = 10$ ms^{-2}) is (a) 1 kgf, (b) 2 kgf, (c) 1·5 kgf, (d) 5 kgf.

Ans. We have $T - 1\times g = 1\times 5$ or $T = 15$ $N = 1{\cdot}5$ kgf.

So the answer is (c).

16. Two masses 5 kg and 3 kg are suspended from the ends of an unstretchable light string passing over a frictionless pulley. When the masses are released, the pressure on the pulley is (a) 8 kgf, (b) 2 kgf, (c) 15 kgf, (d) 7·5 kgf.

Ans. Considering 'free-body' diagrams of the masses we have $T - 3g = 3a$ and $5g - T = 5a$. Solving for T we have $T = 15/4\ g$. F (force on the pulley) $= 2T = 7{\cdot}5\ gN = 7{\cdot}5$ kgf.

So the answer is (d).

17. A particle of mass $m = 9\times 10^{-31}$ kg moving to the wall of a vessel at a velocity of $v = 600$ ms^{-1} strikes the wall of the vessel at an angle 60° to the normal and rebounds at the same angle at the same speed. The impulse of the force experienced by the wall during the impact is (a) $5{\cdot}4\times 10^{-27}$ Ns, (b) $5{\cdot}4\times 10^{-28}$ Ns, (c) 9×10^{-28} Ns, (d) 3×10^{-21} Ns.

Ans. Impulse of the force = change in momentum.

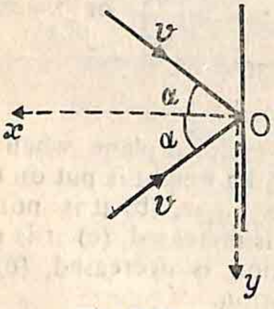


Fig. 2.18

$\vec{v_i}$ (initial velocity) $= -v\cos\alpha\,\hat{i} + v\sin\alpha\,\hat{j}$ with reference to the frame, shown in the figure;

$\vec{v_f}$ (final velocity) $= v\cos\alpha\,\hat{i} + v\sin\alpha\,\hat{j}$;

$\therefore \Delta\vec{v} = \vec{v_f} - \vec{v_i} = 2v\cos\alpha\,\hat{i}$;

$\therefore \Delta v = \sqrt{2}\cos\alpha$,

$\therefore$ Change in momentum

$= m\Delta v = 2mv\cos\alpha$

$= 2\times 9\times 10^{-31}\times 600\times \cos 60°$

$= 5{\cdot}4\times 10^{-28}$ Ns.

So the answer is (b).

18. A 4000 kg rocket is set for firing. If the exhaust speed is 1000 ms^{-1}, the mass to be ejected per second to just overcome gravitational pull ($g=10$ ms^{-2}) is (a) 20 kg, (b) 10 kg, (c) 5 kg, (d) 40 kg.

Sol. We have $F_{ext.}=m\frac{dv}{dt}+u_{rel}\frac{dm}{dt}$. Here $F_{external}=mg$

and $$\frac{dv}{dt}=0; \quad \therefore \quad mg=u_{rel}\frac{dm}{dt}$$

or $$\frac{dm}{dt}=\frac{mg}{u_{rel}}=\frac{4000g}{1000}=4g=40.$$

So the answer is (d).

19. A truck, weighing 8000 kg, is moving along a track with negligible friction at 1·8 ms^{-1} with the shutdown engine when it begins to rain hard. The raindrops fall vertically with respect to the ground. The speed of the truck when it has collected 1000 kg of rain is

(a) 1·6 ms^{-1}, (b) 10 ms^{-1}, (c) 3 ms^{-1}, (d) 9 ms^{-1}.

Ans. The system (truck+raindrops) is free from enternal horizontal force and so their momentum along the horizontal is conserved.

$$\therefore \quad 8000\times 1{\cdot}8=9000\,v' \quad \text{or} \quad v'=1{\cdot}6 \text{ ms}^{-1}.$$

So the answer is (a).

20. A spring of force constant k is cut into three equal pieces. The force constant of each part is

(a) k, (b) $\frac{2k}{3}$, (c) $3k$, (d) $\frac{k}{3}$.

Ans. The force constant is inversely proportional to the length. Here length is decreased three times and so the force constant is increased three times. So the answer is (c).

NUMERICL QUESTIONS

1. A 100 kph wind blows normally against one wall of a house having an area of 50 m^2. Calculate the force exerted on the wall if the air moves parallel to the wall after striking it and has a density 1·134 kgm^{-3}. [JEE (W. Bengal) 1981]

Sol. Let us first deduce a general formula.

Change in velocity of air along the normal to the wall $=v-0=v$. (Velocity along the normal after the air strikes the wall is zero because it moves parallel to it.)

Amount of air striking the wall per second $=A\times v$, where $A=$ area of wall.

Mass of air striking the wall per second $= A \times v \times \rho$,
where ρ = density of air.

$\therefore$ Change in momentum per second $= (A.v.\rho).v = Av^2\rho$.

By laws of motion the wall exerts this force and the air exerts the same force on the wall.

$\therefore$ Force exerted by the air on the wall $= Av^2\rho$

$$= 50 \times \left(\frac{100 \times 1000}{3600}\right)^2 \times 1{\cdot}134 = 4{\cdot}375 \times 10^4 \text{ N.} \quad \textit{Ans.}$$

2. A lift is going up. The total mass of passengers is 1500 kg. The variation of the speed of the lift is as shown in the graph. (a) What will be the tension in the rope pulling the lift at 't' equal to (i) 1 s, (ii) 6 s, (iii) 11 s?

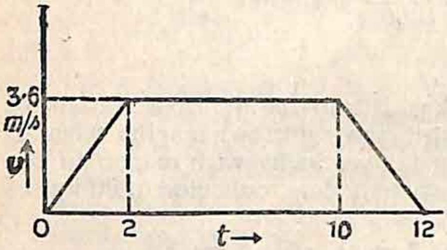


Fig. 2.19

(b) What is the height to which the lift takes the passengers?

(c) What is the average speed and average acceleration during the entire motion? **(IIT 1976)**

Sol. As seen in the figure the lift has uniform acceleration from $t = 0$ to $t = 2$ s. Acceleration = slope $= \frac{3{\cdot}6}{2} = 1{\cdot}8$ ms^{-2}.

Let us first deduce a general expression for the tension T.

The forces acting on the lift are (1) tension T upward, (2) gravitational pull mg downward.

$$\therefore \quad T - mg = ma \quad \text{or} \quad T = m(a+g) \text{ newton.}$$

(i) At $t = 1$ s, $a = 1{\cdot}8$ ms^{-2}. $\therefore$ $T = 1500(1{\cdot}8 + 9{\cdot}8) = 17400$ N. *Ans.*

(ii) At $t = 6$ s, a = slope = 0. $\therefore$ $T = 1500 \times 9{\cdot}8 = 14700$ N. *Ans.*

(iii) At $t = 11$ s, a = slope $= -1{\cdot}8$ ms^{-2}.

$\therefore$ $T = 1500(9{\cdot}8 - 1{\cdot}8) = 12000$ N. *Ans.*

(b) Height = distance covered by the lift = area under v-t curve = area of the trapezium $= \frac{1}{2} \times (12 + 8) \times 3{\cdot}6 = 36$ m. *Ans.*

(c) Average speed $= \frac{\text{total distance}}{\text{total time}} = \frac{36}{12} = 3$ ms^{-1}. *Ans.*

Average acceleration $= \frac{v_f - v_i}{t} = \frac{0-0}{12} = 0$. *Ans.*

3. A homogenous rod with a length L is acted upon by two forces F_1 and F_2 applied to its ends and directed oppositely. With

what force F will the rod be stretched in the cross-section at a distance l from the end where F_1 is applied ?

Sol. Suppose $F_2>F_1$. Then motion of the rod will take place in the direction of the force F_2 with acceleration, say, a. Considering motion of the entire rod we have $F_2-F_1=$ net force on the rod $=mL\times a$ where m is the mass of the rod per unit length of it.

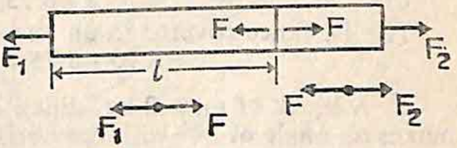


Fig. 2.20

Considering motion of length l from the first end,

$$F-F_1=mla.$$

Dividing, we have $$\frac{F-F_1}{F_2-F_1}=\frac{l}{L}$$

or $$F=\frac{(F_2-F_1)l}{L}+F_1.\quad \textit{Ans.}$$

4. A cart with a mass $M=\frac{1}{2}$ kg is connected by a string to a weight having a mass $m=200$ gm. At the initial moment the cart moves to the left along a horizontal plane at a speed $v_0=7$ ms (see Fig.). Find the magnitude and direction of the cart, the place where it will be and the distance covered by it in $t=5$ s.

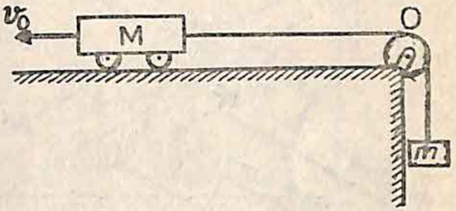


Fig. 2.21

Sol. Let us fix up a frame of reference at the edge of the plane with leftward direction as x-axis. Let x_0 be the distance of the cart at $t=0$. Let a be the common acceleration of m and M. Considering 'free-body' diagrams, we have

$$mg-T=ma \quad \text{and} \quad T=M.a.$$

Adding, we have $$mg=(M+m)a$$

or $$a=\frac{mg}{M+m}=\frac{\cdot 2\times 9\cdot 8}{\cdot 5+\cdot 2}=2\cdot 8\ \text{ms}^{-2}.$$

Since M has initial velocity to the left, a is in fact retardation for M.

$$\therefore \quad v=7+(-2\cdot 8)\times 5=-7\ \text{ms}^{-1}.$$

That is, after 5 s the cart will have the same speed but in the opposite direction. *Ans*

$$x=x_0+7\times 5+\tfrac{1}{2}(-2\cdot 8)\times 5^2=x_0+35-35=x_0.$$

That is, the cart lies in the same position after 5 s. *Ans.*

Let the cart go by s to the left from its initial position. Then from the third kinematic ($v^2 - v_0^2 = 2as$),

$$0^2 - 7^2 = 2(-2{\cdot}8)\ s$$

or $$s = 8{\cdot}75\ \text{m}.$$

$\therefore$ Distance covered in up and down journey

$$= 8{\cdot}75 + 8{\cdot}75 = 17{\cdot}5\ \text{m}. \quad \textit{Ans.}$$

5. A block of mass 2 kg slides down an inclined plane which makes an angle of 30° with the horizontal. The coefficient of friction between the block and the surface is $\sqrt{\frac{3}{2}}$.

(i) What force must be applied to the block so that the block moves down the plane without acceleration ?

(ii) What force should be applied to the block so that it can move up without any acceleration ?

(iii) Calculate the ratio of the powers needed in the above two cases, if the block moves with the same speed in both the cases.

(IIT 1976)

Sol. Make a 'free-body' diagram of the block. Take the force of friction opposite to the direction of motion.

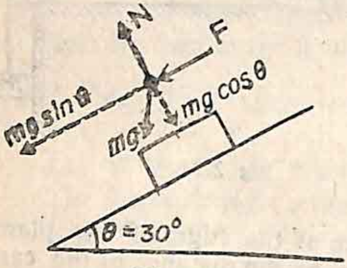


Fig. 2.22

(i) Projecting forces along and perpendicular to the plane

$$N = mg\cos\theta.$$

$F + mg\sin\theta - f = 0$ (since there is no acceleration)

or $F + mg\sin\theta - \mu N = 0$

or $F + mg\sin\theta = \mu mg\cos\theta$

or $F = \mu mg\cos\theta - mg\sin\theta$

$= mg(\mu\cos\theta - \sin\theta)$

$= 2\times 9{\cdot}8(\sqrt{\tfrac{3}{2}}.\cos 30° - \sin 30°)$

$= 19{\cdot}6\left(\sqrt{\dfrac{3}{2}} \times \dfrac{\sqrt{3}}{2} - \dfrac{1}{2}\right)$

$$= 19{\cdot}6\left(\frac{3}{2\sqrt{2}} - \frac{1}{2}\right) = 11{\cdot}0 \text{ newtons.} \quad \textit{Ans.}$$

(ii) This time the direction of F is reversed and that of the frictional force is also reversed.

$$\therefore \quad N = mg\cos\theta; \ F = mg\sin\theta + f$$

or $$F = mg(\mu\cos\theta + \sin\theta) = 19{\cdot}6\left(\frac{3}{2\sqrt{2}} + \frac{1}{2}\right) = 30{\cdot}6 \text{ newtons.}$$

(iii) Power = force × velocity.

Since the velocity is the same in both the cases, ratio of the powers in the two cases is equal to the ratio of the forces.

$$\therefore \quad \frac{P_1}{P_2} = \frac{F_1 v}{F_2 v} = \frac{F_1}{F_2} = \frac{11{\cdot}0}{30{\cdot}6}. \quad \textit{Ans.}$$

6. In the system shown in the figure, m_1 slides down a frictionless inclined plane. The pulleys are massless and frictionless. Find the accelerations of m_1 and m_2 and the tensions of strings.

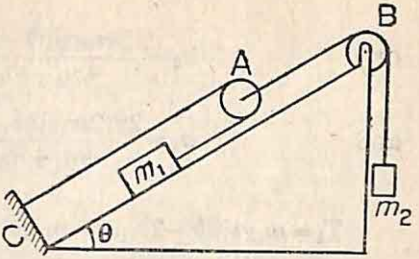


Fig. 2.23

Sol. Let m_2 go up by x when m_1 goes down the plane by y. Here the masses move under the constraints that the lengths of the strings do not change. Since the length of the string connecting m_2 with the pulley A does not change, displacement of A down the plane must be same as the upward displacement of m_2. Let l_1 and l_2 be the initial lengths of the horizontal portions of the string connecting m_1 to the fixed support round the pulley A. After displacements of A and m_1 down the plane l_1 becomes l_1-x and l_2 becomes l_2-x+y. Since the length of the string does not change, neither the string slackens,

$$l_1+l_2=l_1-x+l_2-x+y$$

or $2x=y$.

Differentiating with respect to t,

$$2\frac{dx}{dt}=\frac{dy}{dt}$$

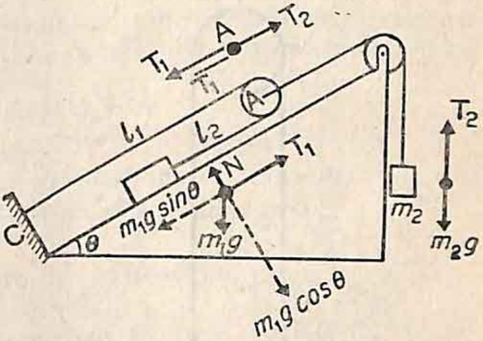


Fig. 2.24

or $\quad 2\dot{x}=\dot{y}\quad$ or $\quad 2v_2=v_1$

and $\quad 2\ddot{x}=\ddot{y}\quad$ or $\quad 2a_2=a_1$.

From the free-body diagram of m_2,

$$T_2-m_2g=m_2a_2. \qquad \text{.. (i)}$$

From the free-body diagram of m_1,

$$m_1g\sin\theta-T_1=m_1a_1$$

or

$$m_1g\sin\theta-T_1=2m_1a_2. \qquad \text{.. (ii)}$$

From the free-body diagram of A,

$$T_2-2T_1=0\times a_2 \quad (\because \ A \text{ is massless})$$

or

$$T_2=2T_1. \qquad \text{.. (iii)}$$

$$\therefore\ 2T_1-m_2g=m_2a_2. \qquad \text{.. (i)}$$

$$m_1g\sin\theta-T_1=2m_1a_2. \qquad \text{.. (ii)}$$

Eliminating T_1, we have

$$2m_1g\sin\theta-m_2g=(4m_1+m_2)a_2$$

$$\left.\begin{aligned} \text{or} \quad a_2 &= \frac{2m_1 g\sin\theta - m_2 g}{4m_1 + m_2} \\ \text{and} \quad a_1 &= \frac{2g(2m_1\sin\theta - m_2)}{4m_1 + m_2}. \end{aligned}\right\} \textit{Ans.}$$

$$T_1 = m_1 g\sin\theta - 2m_1 a_2 = m_1 g\sin\theta - 2m_1 \times \frac{2m_1 g\sin\theta - m_2 g}{4m_1 + m_2}$$

$$\left.\begin{aligned} \text{or} \quad T_1 &= \frac{m_1 m_2 g(2 + \sin\theta)}{4m_1 + m_2} \\ \text{and} \quad T_2 &= \frac{2m_1 m_2 g(2 + \sin\theta)}{4m_1 + m_2}. \end{aligned}\right\} \textit{Ans.}$$

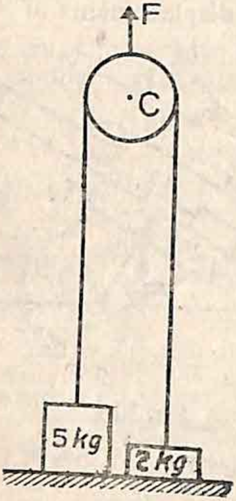


Fig. 2.25

7. Two blocks of masses 5 kg and 2 kg (see figure) are initially at rest on the floor. They are connected by a light string, passing over a light frictionless pulley. An upward force F is applied on the pulley and maintained constant. Calculate the accelerations a_1 and a_2 of the 5 kg and 2 kg respectively when F is (i) 10 N, (ii) 50 N, (iii) 110 N ($g = 10$ ms^{-2})

Sol. Here apart from the constraint that the string is unstretchable, additional constraint is that neither of them can go downward. So they will be lifted only when tension of the string will exceed gravitational pull on them.

Now 5 kg = 50 N and 2 kg = 20 N.

(i) Considering free-body diagram of the pulley,

$$10 - 2T = 0 \quad (\because \text{ pulley is massless})$$

or

$$T = 5 \text{ N}.$$

So tension is less than gravitational pull on both the blocks.

So no acceleration is produced in them.

$$\therefore \quad a_1 = a_2 = 0. \quad \textit{Ans.}$$

(ii) Now $50-2T=0$, or $T=25$ N.

So 5 kg weight will not be lifted but 2 kg weight will be lifted.

$$\therefore\ 25-20=2\times u_2, \quad \text{or} \quad a_2=2{\cdot}5\ \text{ms}^{-2}.$$

Thus $a_1=0$ and $a_2=2{\cdot}5\ \text{ms}^{-2}$. *Ans.*

(iii) Now $110-2T=0$, or $T=55$ N.

So both the weights are lifted.

$$55-50=5a_1, \quad \text{or} \quad a_1=1\ \text{ms}^{-2} \quad \textit{Ans.}$$

and $55-20=2a_2$, or $a_1=17{\cdot}5\ \text{ms}^{-2}$. *Ans.*

8. A frictionless block 3 carries two other frictionless blocks 1 and 2 as shown in the fig. connected by a light string passing over a weightless and frictionless pulley. What horizontal force must continuously be applied to block 3 so that m_1 and m_2 do not move relative to 3 ?

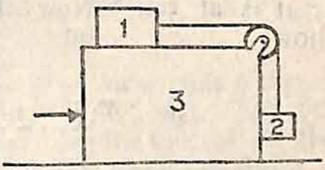


Fig. 2.26

Sol. METHOD I. Let a_1, a_2, a_3 be absolute horizontal accelerations of 1, 2, and 3 to the right and b_1, b_2, b_3 be their downward accelerations.

Obviously $b_1=0$ and $b_3=0$.

a_{13} = relative acceleration of 1 relative to $3=a_1-a_3$.

This is required to be zero by the question. $\therefore\ a_1=a_3$.

b_{23} = relative downward acceleration of 2 relative to 3

$=b_2-b_3=b_2-0=b_2$.

This is also required to be zero by the question.

$$\therefore\ b_2=0,$$

a_{23} = relative acceleration of 2 relative to $3=a_2-a_3=0$ ($\because$ there is no relative displacement of 2 relative to 3 to the right).

$$\therefore\ a_2=a_3.$$

Thus $a_1=a_2=a_3$ and $b_1=b_2=b_3=0$.

Considering 'free-body' diagrams of 1, 2, and 3,

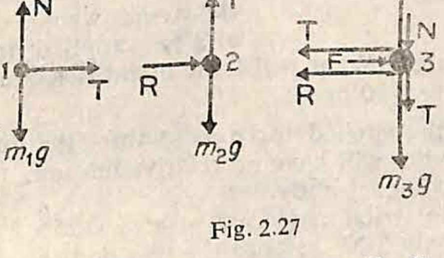


Fig. 2.27

$$T=m_1a_1 \qquad T-m_2g=0 \qquad F-R-T=m_2a_3=m_3a_1$$

$$N=m_1g \qquad R=m_2a_2=m_2a_1 \qquad N'=m_3g+N+T.$$

$$\therefore\quad F=(m_1+m_2+m_3)a_1=(m_1+m_2+m_3)\times\frac{m_2g}{m_1}.\quad \textit{Ans.}$$

Note. Here equations involving normal reactions are not of any use because surfaces are frictionless. These equations will come in use where there is between surfaces.

METHOD II. By D'Alembert's principle.

Let a be the acceleration of the system to the right. Then $F=a(m_1+m_2+m_3)$.... by the second law of motion.

Now let us take the formalism of D'Alembert's principle. We may now consider as if the block m_3 is not moving with acceleration but it is at rest. Now the 'free-body' diagrams of 1 and 2 are as follows :

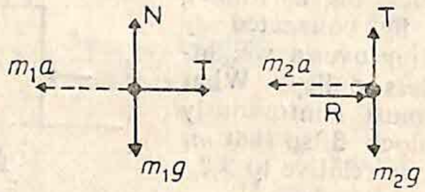


Fig. 2.28

$T-m_1a=m_1a_{13}=0$ ($\because$ $a_{13}=0$) $\qquad R-m_2g=m_2a_{23}=0$

$N-m_1g=0$ $\qquad$ and $\quad m_2g-T=m_2b_{23}=0.$

$$\therefore\quad T=m_1a=m_2g,\quad\text{or}\quad a=\frac{m_2g}{m_1}.$$

$$\therefore\quad F=(m_1+m_2+m_3)\times\frac{m_2g}{m_1}.\quad \textit{Ans.}$$

Note. Note here how the formalism of D'Alembert's principle simplifies the solution of a dynamic problem..

9. A block weighing 20 kg is placed on a smooth surface. A weight of 2 kg is mounted on the block. The coefficient between the block and the weight is 0·25. Calculate the acceleration of the block and the weight and also the frictional force between the block and the weight when a horizontal force of 2 N is applied to the weight as shown in the figure. What will these quantities be if the horizontal force is 20 N ? ($g=10$ ms^{-2})

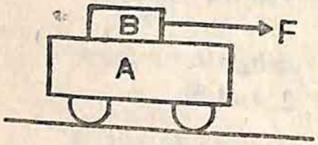


Fig. 2.29

Sol. So long the applied force is within the limiting frictional force, the two bodies will have no relative motion, that is, the two will behave like compact body.

Here limiting frictional force $=\mu mg=0{\cdot}25\times2\times10=5$ N. The applied force is only 2 N. Hence the two bodies will remain intact and will have a common acceleration given by

$$2=(20+2)a,\quad\text{or}\quad a=\tfrac{1}{11}={\cdot}09\ \text{ms}^{-2}.\quad \textit{Ans.}$$

The frictional force = The applied force = 2 N. *Ans.*

When $F=20$ N, the applied force is in excess of the limiting frictional force. So there will be relative motion between the two. The frictional force = the limiting frictional force = 5 N. *Ans.*

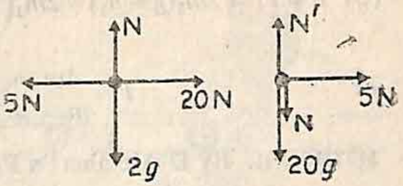


Fig. 2.30

From the free-body diagram of the weight and the block,

$$20-5=2a_1, \quad \text{or} \quad a_1=7{\cdot}5\ \text{ms}^{-2}. \quad \textit{Ans.}$$

$$5=20a_2, \ a_2=0{\cdot}25\ \text{ms}^{-2}. \quad \textit{Ans.}$$

10. A pulley fixed to the ceiling of an elevator car carries a thread whose ends are attached to the loads m_1 and m_2 $(m_1>m_2)$. The car starts going up with an acceleration a. Assuming the masses of the pulley and the thread, as well as the friction, to be negligible find :

(a) the acceleration of m relative to the elevator shaft and relative to the car.

(b) the force exerted by the pulley on the ceiling of the car.

Sol. METHOD I. Let l_1 and l_2 be the lengths of the two portions at the initial moment.

Let in time t, m_1 go down by x_1, m_2 go up by x_2 and the pulley go up by x_3. Then l_1 becomes $l_1+x_1+x_3$ and l_2 becomes $l_2-x_2+x_3$.

$$\therefore \quad l_1+l_2=l_1+x_1+x_3+l_2-x_2+x_3$$

$$\text{or} \qquad 2x_3=x_2-x_1$$

$$\text{or} \qquad 2\ddot{x}_2=\ddot{x}_2-\ddot{x}_1$$

$$\therefore \quad 2a=a_2-a_1. \qquad \text{(i)}$$

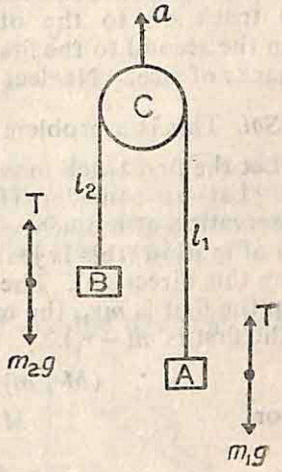


Fig. 2.31

From the free-body diagrams of m_1 and m_2,

$$m_1g-T=m_1a_1$$

$$\text{and} \quad T-m_2g=m_2a_2.$$

$$\text{Adding,} \quad (m_1-m_2)g=m_1a_1+m_2a_2$$
$$=m_1a_1+m_2(2a+a_1)$$

$$\text{or} \quad a_1=\frac{(m_1-m_2)g-2m_2a}{m_1+m_2}. \quad \textit{Ans.}$$

$a_{1\ car}$ = relative acceleration of 1 relative to the car

$$=a_1-(-a)=a_1+a$$

$$\text{or} \quad a_{1\ car}=\frac{(m_1-m_2)g-2m_2a}{m_1+m_2}+a=\frac{m_1-m_2}{m_1+m_2}(g+a). \quad \textit{Ans.}$$

(b) $F=2T=2m_1(g-a_1)=2m_1\left(g-\dfrac{(m_1-m_2)g-2m_2a}{m_1+m_2}\right)$

or $$F=\frac{4m_1m_2}{m_1+m_2}(g+a).\quad Ans.$$

METHOD II. **By D'Alembert's Principle.**

By the formalism indicated by D'Alembert's principle we can assume the pulley to be at rest by imposing inertia forces m_1a and m_2a downward on m_1 and m_2 respectively. Then

$$(m_1g+m_1a)-T=m_1a_{1\,car}=m_1(a_1+a)$$
$$T-m_2(g+a)=m_2a_{2\,car}=m_2(a_1-a)$$
$$(\because\ a_1=a_2 \text{ when the car is supposed to be at rest}).$$

Adding, $m_1(g+a)-m_2(g+a)=(m_1+m_2)a_1-a(m_1-m_2)$

or $$a_1=\frac{(m_1-m_2)g-2m_2a}{(m_1+m_1)}.$$

11. Two identical trucks, each of mass M (excluding sack of rice) move on national highways with speeds v_1 and v_2 towards each other. When they meet each other a sack of rice of mass m is thrown from one truck on to the other and an identical sack of rice is thrown from the second to the first. Calculate their velocities after exchange of sacks of rice. Neglect friction of the road.

Sol. This is a problem on the conservation of momentum.

Let the first truck move to the right and the second truck to the left. Let us consider rightward directions as positive. Consider conservation of momentum of (truck+sack of rice) along the direction of motion (this is justified because there is no external force along this direction). The momentum carried away by a sack thrown from the first is mv_1, the momentum brought in by the second sack to the first is $m(-v_2)$.

$$\therefore\quad (M+m)v_1-mv_1+m(-v_2)=(M+m)v_1'$$

or $$Mv_1-mv_2=(M+m)v_1'$$

or $$v_1'=\frac{M}{M+m}\,v_1-\frac{m}{M+m}\,v_2.\quad Ans.$$

Similarly considering conservation of momentum of the second truck+sacks,

$$v_2'=\frac{M}{M+m}\,v_2-\frac{m}{M+m}.v_1.\quad Ans.$$

12. A flat car of mass M starts moving to the right due to a cosntant force F (see figure). Sand spills on the flat car from a stationary hopper. The velocity of loading is constant and equal to m kg/s. Find the velocity and acceleration of the car after t seconds.

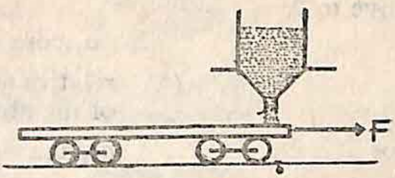


Fig. 2.32

Sol. Instantaneous mass of the car $= M + mt$.

Instantaneous velocity of the car $= v$ (say).

$\therefore$ Instantaneous momentum of the car $= (M+mt)v$.

By Newton's second law,

$$F = \frac{d}{dt}\{(M+mt)v\} = M\frac{dv}{dt} + m\frac{d}{dt}(vt)$$

or $$Fdt = Mdv + md(vt).$$

Integrating, $Ft = Mv + m(vt) + c$ (a constant).

When $t=0,\ v=0.\ \therefore\ c=0.$

$$\therefore\ Ft = Mv + mv.t$$

or $$v = \frac{Ft}{M+mt}.\quad \textit{Ans.}$$

$$a = \frac{dv}{dt} = \frac{F.M}{(M+mt)^2}.\quad \textit{Ans.}$$

13. A block of mass m_1 is placed on the incline of a smooth wedge of mass m_2. The wedge can move with friction on the horizontal surface of the table. Calculate the accelerations (vertical and horizontal components) of the block and the wedge and their interactions.

Sol. Let us denote the block as body 1 and the wedge as body 2. Further let us use a for horizontal acceleration, b for vertical acceleration of bodies with suffix 1 and 2 that is a_1 and b_1 are the rightward horizontal and vertical downward accelerations of 1 and a_2 and b_2 are the leftward and vertical downward acceleration of 2. Obviously $b_2=0$ as the wedge cannot move downward.

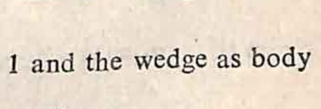


Fig. 2.33

Let a_{12} be the relative acceleration of 1 relative to 2 down the incline. Then $a_{12}\cos\alpha$ is the relative rightward acceleration of 1 relative to 2 and $a_{12}\sin\alpha$ is the downward relative acceleration of 1 relative to 2.

$$\therefore \quad a_{12}\cos\alpha = a_1 - (-a_2)$$

($\because$ relative acceleration is the vector-difference of the absolute accelerations)

or
$$a_{12}\cos\alpha = a_1 + a_2 \quad .. \quad \text{(i)}$$

and
$$a_{12}\sin\alpha = b_1 - b_2 = b_1. \quad .. \quad \text{(ii)}$$

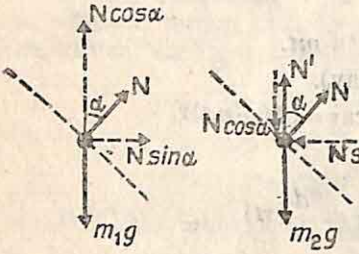


Fig. 2.34

Consider free-body diagram of 1 and 2.

$$N\sin\alpha = m_1a_1$$
$$= m_1(a_{12}\cos\alpha - a_2) \quad .. \quad \text{(iii)}$$
$$m_1g - N\cos\alpha = m_1b_1$$
$$= m_1a_{12}\sin\alpha \quad .. \quad \text{(iv)}$$
$$N\sin\alpha = m_2a_2 \quad .. \quad \text{(v)}$$
$$m_2g - N\cos\alpha = N' = 0.$$

Eliminating a_{12} between (iii) and (iv),

$$m_1g\cos\alpha - N = a_2\sin\alpha. \quad .. \quad \text{(vi)}$$

Substituting the value of N from (v) in (vi),

$$m_1g\cos\alpha - \frac{m_2a_2}{\sin\alpha} = m_1a_2\sin\alpha$$

or
$$a_2 = \frac{m_1g\sin\alpha\cos\alpha}{m_2 + m_1\sin^2\alpha}.$$

From (iii) and (iv),

$$m_1a_1 = m_2a_2.$$

$$\therefore \quad a_1 = \frac{m_2a_1}{m_1} = \frac{m_2g\sin\alpha\cos\alpha}{m_2 + m_1\sin^2\alpha}.$$

From (i), $a_{12}\cos\alpha = a_1 + a_2 = \dfrac{(m_1+m_2)g\sin\alpha\cos\alpha}{m_2 + m_1\sin^2\alpha}.$

$$\therefore \quad a_{12} = \frac{(m_1+m_2)g\sin\alpha}{m_2 + m_1\sin^2\alpha}. \quad .. \quad \text{(vii)}$$

From (ii),
$$a_{12}\sin\alpha = b_1.$$

$$\therefore \quad b_1 = \frac{(m_1+m_2)g\sin^2\alpha}{m_2 + m_1\sin^2\alpha}.$$

From (v), $N = \dfrac{m_2a_2}{\sin\alpha} = \dfrac{m_1m_2g\cos\alpha}{m_2 + m_1\sin^2\alpha}.$

Thus, horizontal acceleration of $1=\dfrac{m_2 g\sin\alpha\cos\alpha}{m_2+m_1\sin^2\alpha}$,

$$\text{vertical acceleration of } 1=\frac{(m_1+m_2)g\sin^2\alpha}{m_2+m_1\sin^2\alpha},$$

$$\text{acceleration of 2 (horizontal)}=\frac{m_1 g\sin\alpha\cos\alpha}{m_2+m_1\sin^2\alpha},$$

$$\text{interactions of 1 and } 2=\frac{m_1 m_2 g\cos\alpha}{m_2+m_1\sin^2\alpha}.$$

14. A rocket weighs 15000 kg when it is on the launching pad on the surface of the earth. It is fixed vertically upward and after all the fuel is burnt out, it weighs 5000 kg. The ejection of the fuel gas is at a constant rate of 5 kg/s with a constant velocity of 1700 m/s relative to the rocket.

Calculate : (a) the thrust on the rocket,

(b) the final speed of the rocket after the complete burnout of the fuel (neglect gravity and air resistance).

Sol. (a) Thrust upward $=u=1700\times 5=8500$ N. *Ans.*

(b) We have for rocket motion

$$F_{ext}=m\frac{dv}{dt}+u_{rel}\frac{dm}{dt}.$$

Here $F_{ext}=0$.

$$\therefore\quad m\frac{dv}{dt}=-u_{rel}\frac{dm}{dt}.$$

or

$$dv=-u_{rel}\frac{dm}{m}.$$

$$\therefore\quad v=-u_{rel}\int_{m_0}^{m}\frac{dm}{m}=-u_{rel}\log\frac{m}{m_0}=u_{rel}\log_e\frac{m_0}{m}.$$

$$\therefore\quad v=1700\times\log_e\tfrac{15000}{5000}=1700\log_e 3=1700\times 2{\cdot}3\times{\cdot}4771$$
$$=1{\cdot}87\ \text{km/s.}\quad \textit{Ans.}$$

EXERCISES

1. A metal plate of mass 100 gm is balanced in mid air by throwing 100 balls per second, each of mass 5 gm vertically upwards from below. The balls get rebounded with the same speed with which they stike the plate. Find the velocity with which they strike the plate.

2. A mass A (500 gm) is placed on a smooth table with a string attached to it. The string goes over a frictionless pulley and is connected to another mass B (200 gm). At $t=0$ the mass A is at distance of 200 cm from the end and moving with a speed of 50 cm/sec towards the left. What will be its position and speed at $t=1$ s ?

(IIT 1975)

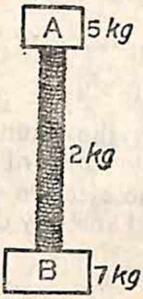


Fig. 2.35

3. Two blocks A and B of masses 5 kg and 7 kg are connected by a heavy rope of 2 kg as shown in fig An upward force of 200 N is applied on A. Find :

(a) acceleration of the system,

(b) tension at the top of the rope,

(c) tension at the mid-point of the rope.

4. An aeroplane requires for take off, a speed of 80 kph, the run on the ground being 100 metres. The mass of the plane is 10,000 kg and the coefficient of friction between the plane and the ground is 0·2. Assume that the plane accelerates uniformly during the take off. What is the minimum force exerted by the engine of the plane for the take off ?

(IIT 1977

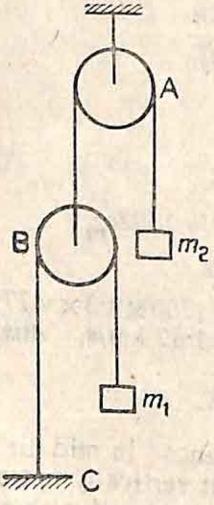


Fig. 2.36

5. In the system shown in the figure, find the accelerations of m_1 and m_2. Take the pulleys and strings as massless and frictionless. Find also the tensions of the strings.

6. Two masses m and $2m$ are connected by a massless string which passes over a light frictionless pulley as shown in the figure. The masses are initially held with equal lengths of the strings of the pulley. Find the velocity of the masses at the instant the lighter mass has moved up a distance of 6·54 m. The string is suddenly cut at that instant. Calculate the time taken by each mass to reach the ground. Acceleration due to gravity = 981 cm/sec^2.

(IIT 1977)

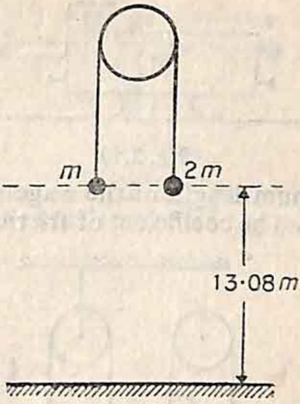


Fig. 2.37

7. A parachute of mass m starts coming down with a constant acceleration a. Determine the ballast mass to be released for parachute to reach the upward acceleration of the same magnitude. The air drag may be neglected.

8. In the arrangement shown in the figure the masses of bodies 1, 2 and 3 are not equal, the masses of the pulley and the threads are negligible, and there is no friction in the pulley. Find the acceleration with which the mass 3 comes down, and the tension of the thread binding together the masses 1 and 2, if the coefficient of friction between these masses and the horizontal surface is equal to k. Take m_1, m_2 and m_3 as the masses of bodies 1, 2 and 3 respectively.

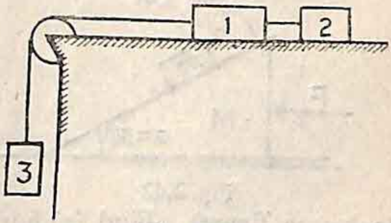


Fig. 2.38

9. A weightless pulley is attached to the apex of two smooth inclined planes forming angle $\alpha = 30°$ and $\beta = 45°$ with the horizontal. Equal weights A and B, 1 kg each, are attached to a light string passing over the pulley. Find (i) the acceleration with which each weight moves, (ii) the tension in the string.

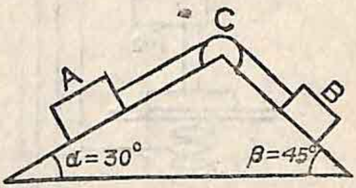


Fig. 2.39

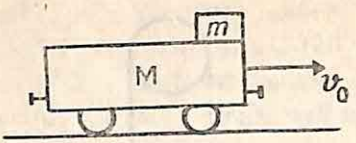


Fig. 2.40

10. A wagon of mass M, length l moves without friction over horizontal rails at a speed of v_0. A small body of mass m is placed on the front edge of the wagon. The initial speed of the body is zero. At what minimum length of the wagon will the body not fall off at the rear end ? The coefficient of friction between the body and the wagon is k.

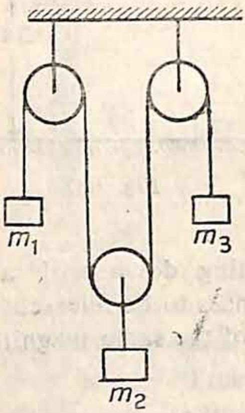


Fig. 2.41

11. A system of two stationary pulleys and a movable pulley is shown in the figure. A string thrown over them carries at its ends weights of masses m_1 and m_3, while a weight of mass m_2 is attached to the axis of the movable pulley. Find the acceleration of each mass. Neglect masses of pulleys and friction at their axis.

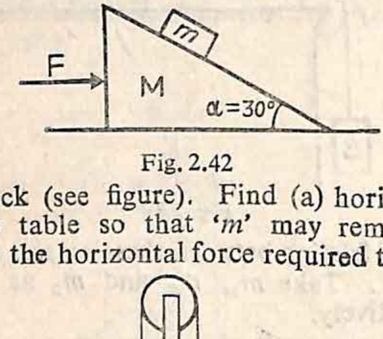


Fig. 2.42

12. A right angled block of mass M rests on a smooth horizontal table with inclination of its hypotenuse plane at 30° to the horizontal. A cubical block of mass 'm' rests on the inclined plane of the block (see figure). Find (a) horizontal acceleration of M relative to the table so that 'm' may remain stationary with respect to M, (b) the horizontal force required to maintain this acceleration.

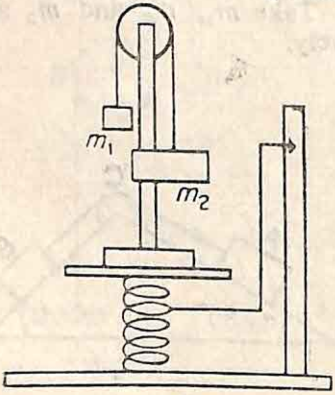


Fig. 2.43

13. In the arrangement shown in the figure weight m_2 slides down the vertical light bar with friction so that both weights move uniformly. Determine the force that the pulley exerts on the bar and the readings of the balance when the weights move. Consider two cases : (i) $m_1=1$ kg, $m_2=3$ kg and (ii) $m_1=3$ kg, $m_2=1$ kg.

14. Two particles of equal masses m and m are connected by a light string of length '$2l$' as shown. A constant force F is applied continuously at the mid-point of the string, always along the perpendicular bisector of the line joining the two particles. Show that when the distance between the particles is '$2x$', the acceleration of approach of the particles is $a = \frac{F}{2m} \cdot \frac{x}{(l^2 - x^2)^{\frac{1}{2}}}$.

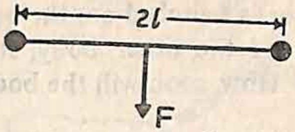


Fig. 2.44

15. Two blocks A and B of masses $m_1 = 4$ kg and $m_2 = 8$ kg are connected as shown in the figure. Find the force P required to move B to the right with uniform speed. Disregard the mass of the pulley and assume it to be frictionless. The coefficient of friction between all surfaces is $\mu = 0{\cdot}25$.

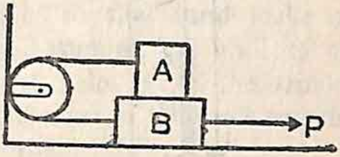


Fig. 2.45

16. Two bars 1 and 2 are placed on an inclined plane forming an angle α with the horizontal (see figure). The masses of the bars are equal to m_1 and m_2, and the coefficient of friction between the plane and the bars are equal to k_1 and k_2 respectively, with $k_1 > k_2$. Find :

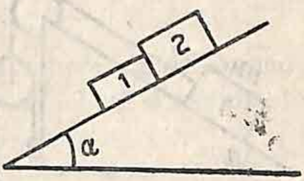


Fig. 2.46

(i) the force of interaction of the bars in the process of motion,

(ii) the minimum value of α at which the bars start sliding down.

17. A small body was projected up an inclined plane, set at $\alpha = 30°$ with the horizontal. Find the coefficient of friction, if the time of ascent of the body is $n = 2$ times less than the time of its descent.

18. A horizontal plane with the coefficient of friction k supports two bodies A and B. B carries an electric motor with a battery. A thread attached to A is wound on the shaft of the motor as shown in

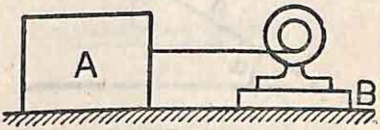


Fig. 2.47

the figure. The distance between the bar and the motor is l. When the motor is switched on, the body A, whose mass is twice as great as that of the other body, starts moving with a constant acceleration a. How soon will the bodies collide ?

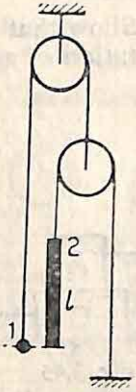


Fig. 2.48

19. In the arrangement shown in the figure the mass of ball is $n = 1{\cdot}8$ times as great as that of the rod 2. The length of the latter is $l = 1$ m. The masses of the pulleys and threads, as well as the friction, are negligible. The ball is set on the same level as the lower end of the rod and then released. How soon will the ball be opposite to the other end of the rod ?

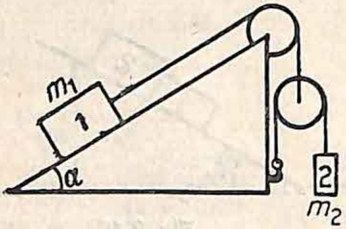


Fig. 2.49

20. Find the acceleration a_2 of body 2 of mass m_2 in the arrangement shown in the figure. The angle of inclination of the plane with the horizontal is equal to α and the masses of the pulleys and the threads, as well as friction are assumed to be negligible. The mass of the body on the inclined plane is m_1 and m_1 is n times less than m_2.

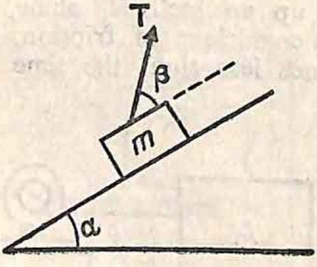


Fig. 2.50

21. A bar of mass m is pulled up with constant speed by means of a thread up an inclined plane forming an angle α with the horizontal (see figure). The coeffcient of friction is equal to k. Find the angle β which the thread must form with the inclined plane for the tension of the thread to be minimum. What is it equal to ?

22. A gun of mass M whose barrel makes an angle α with the horizontal fires a shell of mass m. The whole gun is mounted on frictionless track, so that recoil takes place with no resistive forces. Find the recoil velocity of the car, and the absolute velocity of the shell in magnitude and direction if the muzzle velocity of the shell, relative to the barrel is v. The initial velocity of the car carrying the artillery is V.

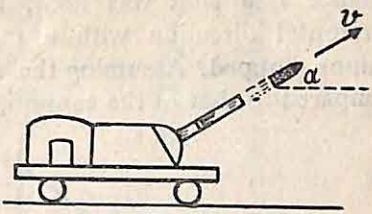


Fig.2.51

23. A truck filled with sand has a hole so that sand leaks out through the bottom at the rate m kg/s. Assuming the pull of the engine constant and it equals F, find the velocity of the truck t seconds after the truck starts moving. The mass of the truck together with sand is M. Neglect friction of the road.

24. At the initial moment a boat is floating in still water of a lake. At the end of each second a ball of mass m is thrown in the horizontal direction with velocity u relative to the boat. Calculate the velocity of the boat in n seconds, disregarding the water resistance.

25. Calculate the coefficient of friction between a floor and a box weighing 1000 kg if a minimum force of 600 kgf is required to start the box moving.

26. In the arrangement shown in the figure, calculate the force that the man must apply on the rope to keep the plane AB in hanging position. The mass of the man is 60 kg and that of the plank 20 kg. Neglect mass of the pulleys and ropes and also friction.

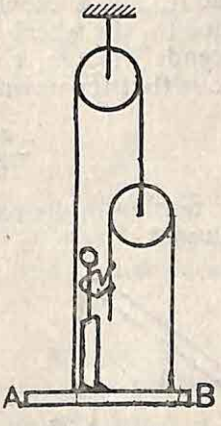


Fig. 2.52

27. A cannon of mass M starts sliding down a smooth inclined plane at an angle α to the horizontal. After the cannon covered a

distance l, a shot was fired, the shell leaving the cannon in the horizontal direction with a momentum p. As a consequence, the cannon stopped. Assuming the mass of the shell to be negligible, as compared to that of the cannon, find the duration of the shot.

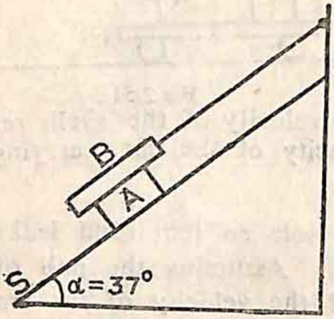


Fig. 2.53

28. A block A, of weight W, slides down an inclined plane S of slope angle 37° at constant velocity while the plank B, also of weight W, rests on top of A. The plank B is attached by a cord to the top of the plane. If the coefficients of kinetic friction is the same between the surfaces A and B and between S and A, determine its value.

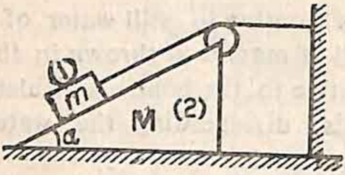


Fig. 2.54

29. In the arrangement shown in the figure the masses m of the bar and M of the wedge of angle α to the horizontal are known, the masses of the pulley and the thread are negligible. The friction is absent. Find the acceleration of the wedge M.

30. A wheel of radius '$4a$' is fastened to an axle of radius a. A weight of mass m_1 is suspended by a rope fastened to the axle and coiled round it, and another weight of mass m_2 is suspended by a rope fastened to the wheel and coiled round its rim in such a way that m_2 descends when m_1 rises and vice-versa. If the system is left to itself, prove that the upward acceleration of m_1 is

$$\frac{4m_2-m_1}{16m_2+m_1}\cdot g$$

and find the tension in the ropes. The masses of the wheel and axle may be neglected.

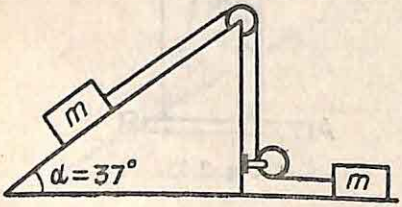


Fig. 2.55

31. Two blocks, each having mass $m=20$ kg, rest on frictionless fixed surfaces as shown. Assuming the pulleys to be light and frictionless, compute the time required for block A to move 1 m down the plane, starting from rest.

32. A flexible rope 10 m long, weighing $\frac{1}{2}$ kg per metre, passes over a small frictionless pulley. It is released from rest with 4 m of the rope hanging from one side and 6 m from the other side of the pulley. (a) What is the acceleration when a length x hangs from the left-hand side, (b) what is the velocity of the rope when it leaves the rope ?

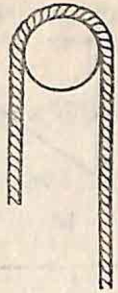


Fig. 2.56

33. A 6000 kg rocket is set for firing. If the exhaust speed is 1 km/s, how much gas must be ejected per second to supply the thrust needed (a) to overcome the weight of the rocket, (b) to give the rocket an initial upward acceleration of $19{\cdot}6\ \mathrm{ms}^{-2}$?

34. Sand drops from a stationary dropper at a rate dM/dt onto conveyor belt moving with velocity $\boldsymbol{v}$. What force and power are needed to keep the belt moving ?

35. A motor boat of mass m moves along a lake with velocity v_0. At the moment $t=0$ the engine of the boat is shut down. Assuming the resistance of water to be proportional to the velocity, that is, $F=-r\ v$ where r is a positive constant, find (a) how long the motorboat moved with the shutdown engine, (b) the distance covered with the shutdown engine till the complete stop.

36. Having gone through a plank of thickness h, a bullet changed its velocity from v_0 to v. Find the time of this motion of the bullet in the plank, assuming resistance to be proportional to square of the velocity.

37. A small bar starts sliding down an inclined plane forming an angle α with the horizontal. The frictional coefficient depends on the distance s covered as $k=as$ where a is a constant. Find the distance covered by the bar till it stops; and find maximum velocity over this distance.

38. A body of mass m rests on a horizontal plane with the coefficient of friction k. At the moment $t=0$, a horizontal force is applied to it, which varies with time as $F=at$ where a is a constant. Find the distance traversed by the body during the first t seconds after the force action began.

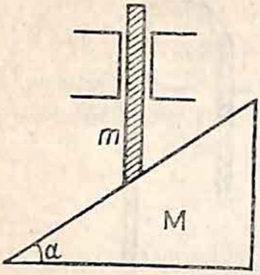


Fig. 2.57

39. In the arrangement shown in the figure find the accelerations of M and m. There is no friction anywhere.

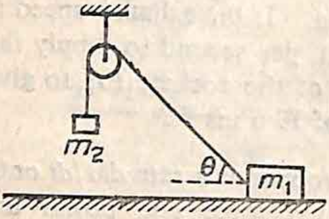


Fig. 2.58

40. A block of mass m_1 rests on a rough horizontal plane with which its coefficient of friction is μ. A light string attached to this block passes over a light frictionless pulley and carries another block of mass m_2. When the system is just about to move, find the value of μ in terms of m_1, m_2 and θ. Also find the tension in the string.

41. A particle A of mass 2 m is held on a smooth horizontal table and is attached to one end of an inelastic string which runs over a smooth light pulley at the edge of the table. At the other end of the string there hangs another particle B of mass m. The distance from A to the pulley is l. The particle A is then projected towards the pulley with velocity u. Find :

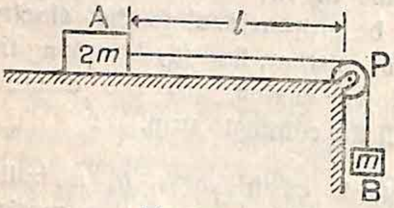


Fig. 2.59

(a) the time before the string becomes taut;

(b) show that after the string becomes taut, the initial velocity of A and B is $\frac{4u}{3}$;

(c) find the common velocity when A reaches the pulley (assume that B has not yet reached the ground).

42. A fixed pulley carries a weightless thread with mass m_1 and mass m_2 at its ends. There is friction between the thread and the pulley. It is found that when m_2 is n times greater than m_1, the thread starts slipping. Find :

(a) the coefficient of friction,

(b) the acceleration of the masses when m_2 is N times m_1 and $N > n$.

43. At the moment $t=0$ a stationary particle of mass m experiences a time dependent force $F=at\ (\tau-t)$, where a is a constant, τ is the time for which force acts. Find :

(a) the momentum of the particle when the action of the force discontinued,

(b) the distance covered by the particle while the force acted.

TEST YOURSELF

Group A

1. Answer the following questions—

(i) Explain why a man getting out of a moving train must run in the same direction for a certain distance.

(ii) It hurts less when an athlete for high jump lands on a heap of sand. Explain.

(iii) An athlete runs over a certain distance before taking a long jump. Explain.

(iv) Ripe mangoes fall down when the tree is shaken. Why ?

(v) A heavy picture frame is suspended by a string passing over a peg in the vertical wall and attached at its two ends to two points in the upper horizontal edge of the frame. Explain that the picture is likely to fall if the string is too short. $3\times5=15$

2. (i) Among the coefficient of static friction, coefficient of kinetic friction and coefficient of rolling friction, the latter is the least. True or false ?

(ii) When we walk we rub our footwear in opposite direction of motion. True or false ?

(iii) Action and reaction are equal and opposite and so they balance each other. True or false ?

(iv) The motion of the centre of mass of a system is determined by external forces only and not the internal forces of the system. True or false ?

(v) Momentum of a system of particles is always conserved. True or false ? $3\times5=15$

3. (i) A ball is suspended by a thread from the ceiling of a tram car. The brakes are applied and the speed of the car changes uniformly from 36 kph to zero in 5 s. The angle by which the ball deviates from the vertical ($g=10\ \text{ms}^{-2}$) is (a) $\tan^{-1}1/3$, (b) $\sin^{-1}1/5$, (c) $\tan^{-1}1/5$, (d) $\cot^{-1}1/3$.

(ii) If you want to stop a car in the shortest distance on an icy road, you should, (a) push hard on the brakes to lock the wheels, (b) push just hard enough to prevent slipping, (c) 'pump' the brakes, (d) not touch the brakes.

(iii) A weight of 10 kg is pulled up a smooth inclined plane forming angle 30° with the horizontal with acceleration 3 ms^{-2}, the tension of the string ($g = 10$ ms^{-2}) is (a) zero, (b) 8 kgf, (c) 7 kgf, (d) 5 kgf.

(iv) A body of mass 1 kg strikes a wall with velocity 8 ms^{-1} along the normal to the wall and returns with 4 ms^{-1}. The impulse of the force experienced by the wall is (a) 4 Ns towards the wall, (b) 4 Ns away from the wall, (c) 12 Ns along the normal towards the wall, (d) 12 Ns along the normal and away from the wall.

(v) A body of mass 10 kg falls from a height 5 m ($g = 10$ ms^{-2}) and stopped within 1/10 s on the ground. The force of interaction is (a) 100 N, (b) zero, (c) 1000 N, (d) 1100 N. $5 \times 4 = 20$

Group B

Answer any five of the following. $5 \times 10 = 50$

4. (a) A cricket ball of mass 150 gm is moving with a velocity of 12 ms^{-1} and is hit by a bat so that the ball is turned back with a velocity of 20 ms^{-1}. The force of the blow acts for 0·01 s on the ball. Find the average force exerted by the bat on the ball.

(b) A homogeneous chain of length l lies on a table. What is the maximum length of the chain that can hang vertically if the coefficient of friction between the chain and the table is μ ?

5. (a) A trolley A of mass M has a small block of mass m in contact with it on its vertical side and held in position by a stop. The trolley is moved to the right by a constant force F and the stop is removed. If the coefficient of friction between A and B is k, calculate the force needed so that the block B is just prevented from falling relative to the trolley.

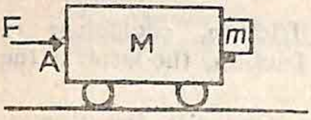


Fig. 2.60

(b) Three blocks of 2 kg, 3 kg and 5 kg are placed in contact on a frictionless table. A constant force of 5 N is applied on 2 kg block towards the other blocks. Calculate the forces of contact between 2 kg and 3 kg and also 3 kg and 5 kg blocks.

6. (a) Two boats move towards each other along parallel paths with different velocities. When they meet, a sack is thrown from one boat onto the other and then an identical sack is thrown from the second onto the first. As a result of this operation, boat 1 stops, and boat 2 keeps on moving in the same direction, with its velocity becoming V. Find the initial velocities of the boats if the mass of each boat (without sack) equal M and the mass of eack sack m. Neglect resistance of water.

(b) Two men, each of mass m, stand on the edge of a stationary car of mass M without men. Assuming the friction to be negligible, find the velocity of the car after both men jump off with the same horizontal velocity u relative to the car : (1) simultaneously, (ii) one after the other. In what case the velocity of car is greater ?

7. A monkey of mass m clings to a rope slung over a light frictionless pulley. The opposite end of the rope is tied to a weight of mass M lying on a smooth horizontal plane (see figure). Find the accelerations of both bodies (relative to the plane) and the tension of the rope for the three cases (neglect mass of pulley and rope) :

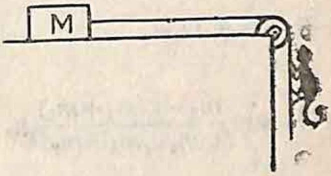


Fig. 2.61

(i) the monkey does not move relative to the rope;

(ii) the monkey moves upwards with acceleration relative to the rope;

(iii) the monkey moves downwards with an acceleration b relative to the rope.

8. A block of mass m is placed on a wedge of mass M (see figure). Find the accelerations of the block and wedge in the reference frame fixed to the table, and the reaction of the wedge on the block. Neglect friction.

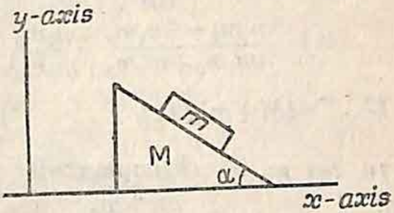


Fig. 2.62

9. A string passing over a light frictionless pulley carries at its ends two variable unequal masses whose sum is constant. If the breaking tension of the string is $\frac{15}{32}$ of the weight of the sum of the masses, show that the least acceleration is g/4 and the least value of the greater masses is 5/8 th of the total mass.

10. The masses of the blocks A and B are m and M. Between A and B there is a constant frictional force F, but B can slide frictionlessly on the horizontal surface. A is set in motion with velocity v_0 while B is at rest. What is the distance moved by A relative to B before they move with the same velocity ?

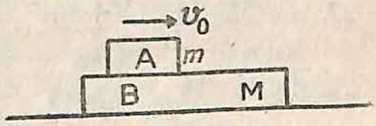


Fig. 2.63

Your score.....

Total marks 100

Answers of Exercises

1. 9·8 m/s.

2. 1·1 m; 1·3 m/s.

3. (a) 4·48 m/s^2, (b) 128·6 N, (c) 114·3 N.

4. $4{\cdot}42 \boxtimes 10^4$ N.

5. $a_1=\dfrac{2(2m_1-m_2)}{4m_1+m_2}g$; $a_2=\dfrac{2m_1-m_2}{4m_1+m_2}\times g$;

$T_1=\dfrac{3m_1m_2}{4m_1+m_2}g$; $T=\dfrac{6m_1m_2}{4m_1+m_2}g$.

6. 2·8 s, ·7 s. **7.** $\dfrac{2ma}{a+g}$.

8. $a=\dfrac{m_3-k(m_1+m_2)}{m_1+m_2+m_3}g$, $T_2=\dfrac{(k+1)m_2m_3}{m_1+m_2+m_3}g$.

9. 1·0 m/s²; 5·9 N. **10.** $\dfrac{Mv_0^2}{2kg(M+m)}$.

11. $a_1=\dfrac{3m_2m_3-4m_1m_3-m_1m_2}{4m_1m_3+m_3m_2+m_2m_1}\times g$, $a_2=\dfrac{m_2m_3-4m_1m_3+m_1m_2}{4m_1m_3+m_1m_2+m_2m_3}g$,

$a_3=\dfrac{3m_1m_2-4m_1m_3-m_2m_3}{4m_1m_3+m_1m_2+m_2m_3}\times g$.

12. $F=(M+m)g\tan\alpha$. **13.** 4 kg. **15.** 5 kgf.

16. (a) $F=\dfrac{(k_1-k_2)m_1m_2g\cos\alpha}{m_1+m_2}$, (b) $\tan\alpha=\dfrac{k_1m_1+k_2m_2}{m_1+m_2}$.

17. ·346. **18.** $t=\sqrt{\dfrac{2l}{3a+kg}}$. **19.** 1·4 s.

20. $a_2=\dfrac{2(2n-\sin\alpha)}{4n+1}\times g$. **21.** $\tan\beta$, $T_{min}=\dfrac{mg(\sin\alpha+k\cos\alpha)}{\sqrt{1+k^2}}$.

22. $u=\dfrac{mv\cos\alpha-(M+m)V}{M+m}$. **23.** $v=\dfrac{Ft}{M-mt}$. **24.** $\dfrac{mu}{M-nm}$.

25. ·75. **26.** 40 kgf. **27.** $\tau=\dfrac{p\cos\alpha-m\sqrt{2gl\cos\alpha}}{Mg\sin\alpha}$.

28. ·25. **29.** $\dfrac{mg\sin\alpha}{M+2m(1-\cos\alpha)}$. **31.** ·82 s; 58·8 N.

32. $\left(1-\dfrac{2x}{l}\right)g$; 6·86 m/s. **33.** (a) 58·8 kg/s, (b) 176·4 kg/s.

34. $F=v\dfrac{dM}{dt}$, $P=v^2\dfrac{dM}{dt}$. **35.** $t=\infty$; $v=v_0-\dfrac{r}{m}x$; $x_{total}=\dfrac{mv_0}{r}$.

36. $t=\dfrac{h(v_0-v)}{v_0v\log\dfrac{v_0}{v}}$, 37. $S=\dfrac{2}{a}\tan\alpha$; $v_{max}=\sqrt{\dfrac{g}{a}\sin\alpha\tan\alpha}$.

38. $S=\dfrac{a}{6m}(t-t_0)^3$. 39. $\dfrac{mg}{m+M\cot^2\alpha}$; $\dfrac{mg\cot\alpha}{m+M\cot^2\alpha}$.

40. $\mu=\dfrac{m_2\cos\theta}{m_1-m_2\sin\theta}$. 41. (a) $t=\dfrac{2u}{g}$, (c) $v=\dfrac{\sqrt{4u^2+6gl}}{3}$.

42. (a) $\mu=\dfrac{1}{\pi}\log n$, (b) $a=\dfrac{N-n}{N+n}g$. 43. (a) $\dfrac{a\tau^3}{6}$, (b) $\dfrac{a\tau^4}{12m}$.

Answers to Test Questions

1. See solutions.
2. (i) True, (ii) True, (iii) False, (iv) True, (v) False.
3. (i) (c), (ii) (b), (iii) (b), (iv) (b), (v) (d).
4. (a) 480 N, (b) $x=\dfrac{\mu l}{\mu+1}$.
5. (a) $F=(M+m)\dfrac{g}{k}$, (b) 4 N, 2·5 N.
6. (a) $v_2=\dfrac{MV}{M-m}$, $v_1=\dfrac{mV}{M-m}$,

   (b) $\dfrac{2mu}{M+2m}$, $\left(\dfrac{mu}{M+m}+\dfrac{mu}{M+2m}\right)$.
7. $\dfrac{mM}{m+M}g$; $\dfrac{mM}{m+M}(g+b)$; $\dfrac{mM}{M+m}(g-b)$.
8. $\dfrac{mg\sin\alpha\cos\alpha}{M+m\sin^2\alpha}$; $\dfrac{Mg\sin\alpha\cos\alpha}{M+m\sin^2\alpha}$; $\dfrac{(M+m)g\sin^2\alpha}{M+m\sin^2\alpha}$; $\dfrac{Mmg\cos\alpha}{M+m\sin^2\alpha}$.

10. $S=\dfrac{mMv_0^2}{2F(m+M)}$.

Solutions to Exercises

1. Let u be the velocity with which each ball strikes the plate.
Change in momentum of each ball $=mu-(-mu)=2mu$.
If n balls hit the plate in every second, then rate of change of momentum of balls is $2mu\times n$.

By Newton's law this is the force exerted by the plate on the balls and the balls exert the same force in the opposite direction. Hence for balance of the plate $2mu \times n = wg$ where w is the mass of the plate.

or $$u = \frac{wg}{2mn} = \frac{100 \times 10^{-3} \times 9{\cdot}8}{2 \times 5 \times 10^{-3} \times 100} = {\cdot}98 \text{ m/s.} \quad \textit{Ans.}$$

2. We have from free-body diagrams of A and B,

$$T = \tfrac{1}{2} \times a$$

and $$\tfrac{1}{5}g - T = \tfrac{1}{5}a.$$

Adding, $\tfrac{1}{5}g = \tfrac{7}{10}a$, or $a = \tfrac{2}{7}g = \dfrac{2 \times 9{\cdot}8}{7} = 2{\cdot}8$ m/s^2.

Since A is moving to the left and the acceleration is to the right, A is first retarded. Let us take leftward direction as the positive direction. Then

S (distance of A from the left end at $t = 1$ s)
$= 2 + ({\cdot}5 \times 1 - \tfrac{1}{2} \times 2{\cdot}8 \times 1^2)$ (formula, $S = S_0 + ut + \tfrac{1}{2}at^2$)
$= 1{\cdot}1$ m. *Ans.*

v (velocity of A at $t = 1$ s) $= {\cdot}5 - 2{\cdot}8 \times 1 = -1{\cdot}3$ m/s.

Thus velocity is 1·3 m/s to the right. *Ans.*

3. (a) Total mass of the system $= 7 + 2 + 5 = 14$ kg.

From the free-body diagram of the system

$$200 - 14 \times 9{\cdot}8 = 14a$$

or $$a = 4{\cdot}48 \text{ m/s}^2. \quad \textit{Ans.}$$

(b) From the free-body diagram of the upper mass A

$$200 - (5g + T_1) = 5 \times 4{\cdot}48$$

or $$T_1 = 200 - 5g - 5 \times 4{\cdot}48 = 128{\cdot}6 \text{ newton.} \quad \textit{Ans.}$$

(c) From the free-body diagram of the portion above the mid-point

$$200 - (6g + T_2) = 6 \times 4{\cdot}48$$

or $$T_2 = 200 - 6g - 6 \times 4{\cdot}48 = 114{\cdot}3 \text{ newton.} \quad \textit{Ans.}$$

4. $$80 \text{ kph} = \frac{80 \times 1000}{3600} = 22{\cdot}2 \text{ m/s.}$$

$$22{\cdot}2^2 = 2a \times 100, \quad \text{or} \quad a = 2{\cdot}46 \text{ m/s}^2.$$

Let P be minimum force of the engine.

Then $$P - \mu mg = ma$$

or $$P = \mu mg + ma = m(\mu g + a)$$

or $$P = 10^4({\cdot}2 \times 9{\cdot}8 + 2{\cdot}46) = 4{\cdot}42 \times 10^4 \text{ newton.} \quad \textit{Ans.}$$

5. Let m_2 go up by x_2 when m_1 goes down by x_1. Denoting the lengths of string on the two sides of B by l_1 and l_2, we have

$$l_1 - x + l_2 + x_1 - x = l_1 + l_2$$

where x is the distance by which the pulley B comes down

or $$x_1 = 2x.$$

Similarly considering 'inextensibility' of the other string around A, $\quad l_1'+x+l_2'-x_2=l_1'+l_2'$

or $\quad x=x_2, \quad \therefore \; x_1=2x_2$

or $\quad \ddot{x}_1=2\ddot{x}_2, \;$ or $\; a_1=2a_2$.

Considering free-body diagrams of m_1, B and m_2, we have

$$m_1g-T_1=m_1a_1,$$

$$2T_1-T_2=0 \text{ (since } B \text{ is massless)}$$

and

$$T_2-m_2g=m_2a_2.$$

Hence

$$\left.\begin{aligned} a_1&=\frac{2(2m_1-m_2)}{4m_1+m_2}\times g\\ a_2&=\frac{2m_1-m_2}{4m_1+m_2}\times g\\ T_1&=\frac{3m_1m_2}{4m_1+m_2}\times g\\ T_2&=\frac{6m_1m_2}{4m_1+m_2}\times g \end{aligned}\right\}. \quad \textit{Ans.}$$

6. a (common acceleration) $=\dfrac{2m-m}{2m+m}\times g=\dfrac{g}{3}=\dfrac{9\cdot81}{3}=3\cdot27$ m/s^2.

Let v be the common velocity acquired by either mass.

$$v^2=2\times3\cdot27\times6\cdot54=6\cdot54^2;$$

$$\therefore \; v=6\cdot54 \text{ m/s}.$$

The velocity of m is upward and it is $(13\cdot08+6\cdot54)$ m above the ground at the time of break-off of the string and the velocity of $2m$ is downward and the height is $(13\cdot04-6\cdot54)$ m above the ground. Denoting the times of arrival by t_1 and t_2, we have

$$-(13\cdot08+6\cdot54)=6\cdot54t_1+\tfrac{1}{2}(-9\cdot81)t_1^2$$

or $\quad -19\cdot62=6\cdot54t_1-\frac{1}{2}\times9\cdot81t_1^2$

or $\quad -6=2t_1-\frac{1}{2}\times3t_1^2, \;$ or $\; 3t_1^2-4t_1-12=0.$

$$t_1=\frac{4\pm\sqrt{16+144}}{6}=2\cdot8 \text{ s.} \quad \textit{Ans.}$$

The negative value is discarded as it is meaningless.

$$-(13\cdot08-6\cdot54)=-6\cdot54t_2+\tfrac{1}{2}(-9\cdot81)t_2^2$$

$$6\cdot54=6\cdot54t_2+\tfrac{1}{2}\times9\cdot81t_2^2$$

or $\quad 2=2t_2+\frac{1}{2}.3t_2^2, \;$ or $\; 3t_2^2+4t_2-4=0$

or $\quad (t_2+2)(3t_2-2)=0, \;$ or $\; t_2=\frac{2}{3}=\cdot7$ s. *Ans.*

7. Let R be the upward thrust on the parachute.

Then $\quad mg-R=ma.$

Let $\triangle m$ be the mass of ballast to be released.

Then $$R-(m-\triangle m)g=(m-\triangle m)a.$$

Adding, $$\triangle mg=2ma-\triangle ma$$

or $$\triangle m=\frac{2ma}{a+g}.\quad Ans.$$

8. We have from 'free-body' diagrams of the masses,

$$m_3g-T_1=m_3a,$$
$$T_1-T_2-km_1g=m_1a,$$
$$T_2-km_2g=m_2a.$$

Adding, $$(m_3-km_1-km_2)g=(m_1+m_2+m_3)a$$

or $$a=\frac{m_3-k(m_1+m_2)}{m_1+m_2+m_3}\times g.\quad Ans.$$

$$T_2=\frac{(k+1)m_2-m_3}{m_1+m_2+m_3}\times g.\quad Ans.$$

9. Let B go down the plane with acceleration a. Then A will go up the plane by the same acceleration as the string is inextensible. Considering motion of the bodies along and perp. to the planes,

$$T-m_1g\sin\alpha=m_1a;\; N_1=m_1g\cos\alpha$$

and $$m_2g\sin\beta-T=m_2a;\; N_2=m_2g\cos\beta.$$

Adding, $$m_2g\sin\beta-m_1g\sin\alpha=(m_1+m_2)a$$

or $$a=\frac{(m_2\sin\beta-m_1\sin\alpha)}{m_1+m_2}\times g$$

$$=\frac{1\sin 45^\circ-1\sin 30^\circ}{1+1}\times g=\frac{\cdot 707-\cdot 5}{2}\times 9{\cdot}8$$

$$=1{\cdot}0\text{ m/s}^2.\quad Ans.$$

$$T=m_1g\sin\alpha+m_1a=1\times 9{\cdot}8\times\cdot 5+1\times 1{\cdot}0$$
$$=4{\cdot}9+1=5{\cdot}9\text{ newton}.\quad Ans.$$

10. The problem can be solved conveniently by the concept of relative displacement, velocity and acceleration.

Initial relative velocity of the mass to the left $=0-(-v_0)=v_0$.

Final relative velocity $=0$ (since both finally move with the same speed).

The same limiting frictional force is effective on m and M; m is accelerated by it and M is retarded.

$$\therefore\quad f_{lim}=kmg=ma_m,\quad\text{or}\quad a_m=kg,$$

Also $$f_{lim}=kmg=Ma_M,\quad\text{or}\quad a_M=\frac{kmg}{M}.$$

Relative acceleration of m relative to M

$$=a_m-(-a_M)=a_m+a_M=kg\left(1+\frac{m}{M}\right).$$

Using $v^2-v_0^2=2as$ we have

$$v_0^2=2\times kg\left(1+\frac{m}{M}\right)s$$

or
$$s=\frac{Mv_0^2}{2kg(M+m)}.$$

Thus '*m*' travels a distances *s* relative to the wagon. If *s* is less than *l*, it will never fall off.

$$\therefore \quad l\geqslant s \quad \text{or} \quad l\geqslant\frac{Mv_0^2}{2kg(M+m)}$$

$$\therefore \quad l_{min}=\frac{Mv_0^2}{2kg(M+m)}. \quad \textit{Ans.}$$

11. Denoting the hanging portions of the string by l_1, l_2, l_2, l_3 and assuming that m_2 goes down and m_1 and m_3 go up we have

$$l_1+l_2+l_2+l_3=l_1-x_1+l_2+x_2+l_2+x_2+l_3-x_3$$

or $0=2x_2-x_1-x_3$ or $2\ddot{x}_2=\ddot{x}_1+\ddot{x}_3$

or
$$2a_2=a_1+a_3.$$

We have from free-body diagrams of m_1, m_2, m_3,

$$T-m_1g=m_1a_1 \quad \text{.. (i)}$$

$$m_2g-2T=m_2a_2 \quad \text{.. (ii)}$$

$$T-m_3g=m_3a_3 \quad \text{.. (iii).}$$

Adding, $(m_2-m_1-m_3)g=m_1a_1+m_2a_2+m_3a_3$ .. (iv).

Subtracting (i) from (iii), $(m_1-m_3)g=m_3a_3-m_1a_1$ (v).

Putting $2a_2=a_1+a_3$ in (iv),

$$2(m_2-m_1-m_3)g=2m_1a_1+m_2(a_1+a_3)+2m_3a_3$$

or
$$2(m_2-m_1-m_3)g=a_1(2m_1+m_2)+a_3(m_2+2m_3) \quad \text{.. (vi).}$$

Eliminating a_3 between (v) and (vi),

$$(m_2+2m_3)(m_1-m_3)g=m_3(m_2+2m_3)a_3-m_1(m_2+2m_3)a_1$$

$$2m_3(m_2-m_1-m_3)g=a_1(2m_1+m_2)m_3+a_3m_3(m_2+2m_3).$$

Subtracting,
$$a_1=\frac{3m_2m_3-4m_1m_3-m_1m_2}{4m_1m_3+m_3m_2+m_2m_1}\times g.$$

Exactly in the same way

$$a_3=\frac{3m_1m_2-4m_1m_3-m_2m_3}{4m_1m_3+m_1m_2+m_2m_3}\times g \quad \text{(replacing 1 by 3 and 3 by 1)}$$

$$a_3=\frac{m_2m_3-4m_1m_3+m_1m_2}{4m_1m_3+m_1m_2+m_2m_3}\times g. \quad \textit{Ans.}$$

12. Let *a* be the acceleration of *m* down the incline relative to *M*.

Then $a\cos\alpha$ = relative horizontal acceleration of m to the right relative to M.

$a\sin\alpha$ = relative vertical downward acceleration of m relative to M.

$\therefore\ a\cos\alpha = a_x - b_x$

where a_x = absolute acceleration of m to the right

b_x = absolute acceleration of M to the right.

$a\sin\alpha = a_y - 0 = a_y$

where a_y = absolute vertical acceleration of m.

Let N be mutual action and reaction between m and M.

Considering horizontal motion of M,

$$F - N\sin\alpha = Mb_x$$

or

$$F - N\sin\alpha = M(a_x - a\cos\alpha) \quad .. \quad \text{(i)}.$$

Considering vertical and horizontal motion of 'm',

$$mg - N\cos\alpha = ma_y = ma\sin\alpha \quad .. \quad \text{(ii)}$$

$$N\sin\alpha = ma_x \quad .. \quad \text{(iii)}.$$

Eliminating a_x between (i) and (iii),

$$F = N\sin\alpha\left(1 + \frac{M}{m}\right) - Ma\cos\alpha \quad .. \quad \text{(iv)}.$$

Eliminating N between (ii) and (iv),

$$a = \frac{F\cos\alpha - mg\sin\alpha\left(1 + \frac{M}{m}\right)}{M + m\sin^2\alpha}.$$

This is the general expression for the acceleration of m down the incline relative to M.

Here $a = 0$. $\therefore\ F\cos\alpha = mg\sin\alpha\left(1 + \dfrac{M}{m}\right)$

or $F = (M+m)g\tan\alpha$. *Ans.*

Alternative method (a simple method)

When $a = 0$, $a_x = b_x$

and $a_y = 0$.

$$\therefore\ F - N\sin\alpha = Mu_x$$

$$mg = N\cos\alpha$$

$$N\sin\alpha = ma_x.$$

$\therefore\ F = (M+m)a_x$ or $a_x = b_x = \dfrac{F}{M+m}$. *Ans.*

Also,

$$F - mg\tan\alpha = M \times \frac{F}{M+m}$$

or

$$F = (M+m)g\tan\alpha. \quad \textit{Ans.}$$

13. Consider motions of m_1 and m_2.

$T-m_1g=0$ (as m_1 and m_2 move with uniform speed there is no acceleration of them)

or $T=m_1g.$

$m_2g=f_{fr}-T=0$ where f_{fr}=frictional force on m_2

or $f_{fr}=(m_2-m_1)g.$

The pressure on the rod $=2T+f_{fr}=2m_1g+(m_2-m_1)g$
$=(m_1+m_1)g=(1+3)g=4$ kgf.

Thus in both the cases reading on the scale is 4 kg. *Ans.*

14. Let N be the reaction of the support on each of the masses. Considering motion of the knot 'O' we have

$$F-2T\cos\theta=0$$

or $F=2T\cos\theta.$

Considering motion of each mass

$$N=T\cos\theta$$

and $T\sin\theta=ma$ or $T=\dfrac{ma}{\sin\theta},$

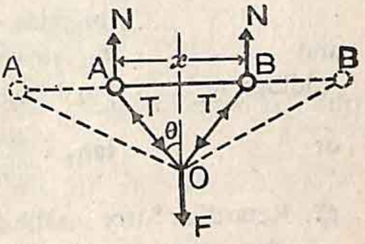


Fig. 2.64

$$\therefore \quad F=2\left(\frac{ma}{\sin\theta}\right)\cos\theta$$

or $F=2ma\cot\theta=2ma\times\dfrac{\sqrt{l^2-x^2}}{x}.$

$$a=\frac{F}{2m}\cdot\frac{x}{\sqrt{l^2-x^2}}.\quad \textit{Proved.}$$

15. Consider 'free-body' diagram of B. Here N is the reaction of the table on B and N' is the reaction of A on B.

We have $N=N'+m_2g$

and $P-\mu N'-\mu N-T=0$

or $P=\mu(N+N')+T$

or $P-T=\mu(N+N-m_2g)$

or $P-T=2\mu N-\mu m_2g.$

From free-body diagrams of A

$T=\mu N'$

and $N'=m_1g$

$\therefore \quad T=\mu m_1g$

and $N=(m_1g+m_2g).$

$\therefore \quad P-\mu m_1g$
$=2\mu(m_1g+m_2g)-\mu m_2g$

$P=3\mu m_1g+\mu m_2g$

$P=\mu(3m_1+m_2)g$
$=\cdot25(3\times4+8)\times g=5g$ newton

$P=5$ kgf. *Ans.*

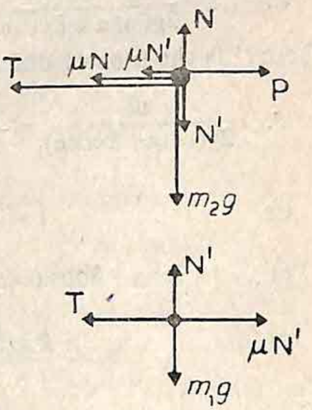


Fig. 2.65

16. Let N_1 and N_2 be the reactions of the plane on 1 and 2 and F is the force of interaction between them.

From free-body diagrams of 1 and 2 we have

$$m_1 g\sin\alpha - k_1 m_1 g\cos\alpha + F = m_1 a$$
$$m_2 g\sin\alpha - k_2 m_2 g\cos\alpha - F = m_2 a.$$

Eliminating a between these two equations

$$F = \frac{(k_1 - k_2) m_1 m_2 g\cos\alpha}{m_1 + m_2}. \quad \textit{Ans.}$$

(b) When the bars just start sliding, $a = 0$.

$$\therefore \quad m_1 g\sin\alpha - k_1 m_1 g\cos\alpha + F = 0$$

and $\quad m_2 g\sin\alpha - k_2 m_2 g\cos\alpha - F = 0.$

Adding, $\quad (m_1 + m_2) g\sin\alpha = (k_1 m_1 + k_2 m_2) g\cos\alpha$

or $\quad \tan\alpha = \dfrac{k_1 m_1 + k_2 m_2}{m_1 + m_2}. \quad$ *Ans.*

17. Retarding force up the plane $= mg\sin\alpha + kmg\cos\alpha$.

$\therefore$ Retardation up the plane $= g\sin\alpha + kg\cos\alpha$.

Let u be the velocity of projection and s is the distance by which the body goes up the plane.

Then $\quad u = (g\sin\alpha + kg\cos\alpha)t$ where t is the time of ascent

and $\quad u^2 = 2(g\sin\alpha + kg\cos\alpha)s \quad$ or $\quad s = \dfrac{u^2}{2g(\sin\alpha + k\cos\alpha)}$.

During the return journey,

force down the plane $= mg\sin\alpha - kmg\cos\alpha$

$\therefore$ acceleration down the plane $= g\sin\alpha - kg\cos\alpha$

$$\therefore \quad s = \frac{u^2}{2g(\sin\alpha + k\cos\alpha)} = \tfrac{1}{2} g(\sin\alpha - k\cos\alpha)t'^2$$

where t' is the time of descent, $t' = nt$ (given).

$$\therefore \quad \frac{u^2}{2g(\sin\alpha + k\cos\alpha)} = \frac{1}{2} g(\sin\alpha - k\cos\alpha) n^2 . \frac{u^2}{g^2(\sin\alpha + k\cos\alpha)^2}$$

or $\quad 1 = \dfrac{n^2(\sin\alpha - k\cos\alpha)}{\sin\alpha + k\cos\alpha}$

or $\quad \sin\alpha + k\cos\alpha = n^2\sin\alpha - kn^2\cos\alpha$

or $\quad k = \dfrac{n^2 - 1}{n^2 + 1}\tan\alpha$

$$= \frac{2^2 - 1}{2^2 + 1}\tan 30^\circ = \tfrac{3}{5} \times {\cdot}577 = {\cdot}346. \quad \textit{Ans.}$$

18. Let a and a' be the accelerations of A and B.

Let T be the tension of the string.

Then
$$T - k2mg = 2ma$$
$$T - kmg = ma'.$$

Subtracting,
$$kmg = ma' - 2ma$$

or
$$kg = a' - 2a.$$

Now $s = \frac{1}{2}at^2$ and $s' = \frac{1}{2}a't^2$

and $s + s' = l.$

$$\therefore \quad \tfrac{1}{2}at^2 + \tfrac{1}{2}a't^2 = l$$

or
$$\tfrac{1}{2}at^2 + \tfrac{1}{2}(kg + 2a)t^2 = l$$

or
$$t = \sqrt{\frac{2l}{3a + kg}}. \quad \textit{Ans.}$$

19. If the ball goes up by x_1, the bar will go down by $2x_1$ during the same interval.

$$\therefore \quad x_2 = 2x_1 \quad \text{or} \quad \ddot{x}_2 = 2\dot{x}_1 \quad \text{or} \quad a_2 = 2a_1.$$

From free-body diagram of the ball
$$T - m_1g = m_1a_1$$

or
$$T = m_1g + m_1a_1 \quad \ldots \text{(i)}.$$

From free-body diagram of the bar
$$m_2g - T' = m_2a_2 = m_22a_1$$

or
$$T' = m_2g - 2m_2a_1.$$

From the free-body diagram of the movable pulley
$$T - 2T' = 0$$

or
$$T = 2T' \quad \text{or} \quad T = 2m_2g - 4m_2a_1 \quad \ldots \text{(ii)}.$$

From (i) and (ii)
$$m_1g + m_1a_1 = 2m_2g - 4m_2a_1$$

or
$$a_1 = \frac{2m_2 - m_1}{m_1 + 4m_2} \times g.$$

Since
$$m_1 = nm_2 \text{ (given)},$$
$$\therefore \quad a_1 = \frac{2 - n}{n + 4}g$$

and
$$a_2 = \frac{2(2 - n)}{n + 4}g.$$

Relative acceleration of 1 relative to 2
$$= a_1 - (-a_2) = a_1 + a_2 = \frac{3(2 - n)}{n + 4}.g$$

Initial relative velocity $= 0.$

Relative displacement of 1 relative to $2 = l$.

$$\therefore \quad l = 0 \times t + \tfrac{1}{2}.\frac{3(2-n)}{n+4}gt^3 \qquad \text{(from } s = v_0t + \tfrac{1}{2}at^2\text{)}$$

$$\therefore \quad t = \sqrt{\frac{2l(n+4)}{3g(2-n)}}$$

$$= \sqrt{\frac{2 \times 1(1{\cdot}8+4)}{3 \times 9{\cdot}8(2-1{\cdot}8)}} = \sqrt{\frac{2 \times 5\,8}{3 \times 9{\cdot}8 \times {\cdot}2}} = 1{\cdot}40 \text{ s.} \quad \textit{Ans.}$$

20. From the fact that the lengths of the strings are constants, it follows that when m_1 is pulled up the plane by x_1, m_2 goes down by $2x_1$.

$$\therefore \quad x_2 = 2x_1 \quad \text{or} \quad \ddot{x}_2 = 2\ddot{x}_1 \quad \text{or} \quad a_2 = 2a_1.$$

From the free-body diagram of m_1

$$T - m_1g\sin\alpha = m_1a_1 = m_1 \times \frac{a_2}{2}$$

or

$$2T - 2m_1g\sin\alpha = m_1a_2 \qquad \text{.. (i)}$$

From the free-body diagram of m_2

$$m_2g - T' = m_2a_2.$$

From the free-body diagram of the massless movable pulley

$$T = 2T'$$

$$\therefore \quad m_2g - \frac{T}{2} = m_2a_2$$

or

$$2m_2g - T = 2m_2a_2. \qquad \text{.. (ii)}$$

Eliminating T between (i) and (ii)

$$a_2 = \frac{4m_2 - 2m_1\sin\alpha}{m_1 + 4m_2} \times g.$$

It is given that $m_3 = nm_1$

$$\therefore \quad a_2 = \frac{4n - 2\sin\alpha}{1+4n} \times g$$

or

$$a_2 = \frac{2(2n - \sin\alpha)}{4n+1} \times g. \quad \textit{Ans.}$$

21. Considering motion of the mass along and perp. to the plane we have

$$T\cos\beta - mg\sin\alpha - kN = 0$$

$$T\sin\beta + N = mg\cos\alpha$$

$$\therefore \quad N = mg\cos\alpha - T\sin\beta$$

$$\therefore \quad T\cos\beta - mg\sin\alpha - k(mg\cos\alpha - T\sin\beta) = 0$$

or $$T=\frac{mg\sin\alpha+kmg\cos\alpha}{\cos\beta+k\sin\beta}.$$

Put $$k=\tan\lambda.$$

Then $$T=\frac{mg\sin(\alpha+\lambda)}{\cos(\beta-\lambda)}.$$

Obviously T is minimum when $\cos(\beta-\lambda)$ is maximum, that is, $+1$.

$\therefore$ T is minimum when $\cos(\beta-\lambda)=1$.

or $$\beta-\lambda=0, \text{ or } \beta=\lambda.$$

Thus T is minimum when $k=\tan\beta$. *Ans.*

$$T_{min}=mg\sin(\alpha+\beta)=mg(\sin\alpha\cos\beta+\cos\alpha\sin\beta)$$

or $$T_{min}=mg\left(\sin\alpha.\frac{1}{\sqrt{1+k^2}}+\cos\alpha\frac{k}{\sqrt{1+k^2}}\right)$$

or $$T_{min}=\frac{mg(\sin\alpha+k\cos\alpha)}{\sqrt{1+k^2}}. \quad \textit{Ans.}$$

22. The horizontal component of the relative velocity of the bullet $=v\cos\alpha$.

Let the absolute horizontal velocity of the bullet be v'.

Then $v\cos\alpha=v'-(-u)$ where u is the velocity of recoil of the car

or $$v'=v\cos\alpha-u.$$

From the law of conservation of momentum along the horizontal (the system is free from external force along the horizontal)

$$(M+m)V=-Mu+m(v\cos\alpha-u)$$

from which

$$u=\frac{mv\cos\alpha-(M+m)V}{M+m}. \quad \textit{Ans.}$$

23. Instantaneous mass of the truck $=(M-mt)$.

Instantaneous velocity of the car $=v$ (say).

Instantaneous momentum of the car $=(M-mt)v$.

By Newton's second law, $F=\frac{d}{dt}\{(M-mt)v\}$

or $$Fdt=Mdv-md(vt).$$

Integrating, $$Ft=Mv-mvt+c \text{ (a constant)}$$

At $t=0$, $v=0$; $\therefore$ $c=0$.

$$\therefore \quad Ft=Mv-mvt$$

$$v=\frac{Ft}{M-mt}. \quad \textit{Ans.}$$

24. Let us denote the velocity of the boat in rth second by v_r to the left (assuming that balls are being thrown to the right) and that in $(r+1)$th second by v_{r+1} to the left. Let u' be the absolute velocity of the ball to the right when it is thrown at the end of rth second and beginning of $(r+1)$th second.

Then u (relative velocity of ball relative to the boat)

$$=u'-(-v_r)=u'+v_r$$

or
$$u'=u-v_r.$$

Considering 'conservation of momentum to the left'

$$(M-rm)v_r=(M-rm-m)v_{r+1}-m(u-v_r)$$

or
$$(M-rm)v_r=(M-rm-m)v_{r+1}-mu+mv_r$$

or
$$(M-rm-m)v_r=(M-rm-m)v_{r+1}-mu$$

or
$$v_{r+1}-v_r=\frac{mu}{M-(r+1)m}.$$

Putting $r=0, 1, 2, 3, 4\ldots,$

$$v_1-v_0=\frac{mu}{M-m}$$

$$v_2-v_1=\frac{mu}{M-2m}$$

$$v_3-v_2=\frac{mu}{M-3m}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$v_n-v_{n-1}=\frac{mu}{M-nm}$$

Add,
$$v_n=v_0+\frac{mu}{M-nm}.$$

Since the boat is initially at rest, $v_0=0$.

$$\therefore\quad v_n=\frac{mu}{M-nm}.\quad \textit{Ans.}$$

25. Let m be the mass of the box and P is the force applied at an angle α to the horizontal.

Considering horizontal and vertical motion of the box we have

$$P\cos\alpha-\mu N=0$$

and
$$P\sin\alpha+N-mg=0$$

$$\therefore\quad P\cos\alpha=\mu N=\mu(mg-P\sin\alpha)$$

or
$$P=\frac{\mu mg}{\cos\alpha+\mu\sin\alpha}.$$

Put
$$\mu=\tan\lambda.$$

Then
$$P=\frac{\mu mg\cos\lambda}{\cos(\alpha-\lambda)}.$$

Obviously P is minimum when $\cos(\alpha-\lambda)$ is maximum, that is, 1,

or
$$\alpha=\lambda.$$

$$\therefore\quad P_{min}=\mu mg\cos\lambda=\mu mg.\frac{1}{\sqrt{1+\mu^2}}$$

or
$$P_{min}=\frac{\mu mg}{\sqrt{\mu^2+1}}$$

$$\therefore\quad 600g=\frac{\mu\times 1000\times g}{\sqrt{u^2+1}}$$

or $6\sqrt{\mu^2+1}=10\mu$

or $36(\mu^2+1)=100\mu^2$, or $36=64\mu^2$

or $\mu=\frac{6}{8}=\cdot75$. *Ans.*

26. Let T_1 and T_2 be the tensions of the two strings.

Consider 'plank+man' as a system.

The forces acting on this system are : mg, downward and $(T_1+T_2+T_2)$ upwards.

$$\therefore\quad mg=T_1+T_2+T_2. \qquad \text{.. (i)}$$

Considering 'free-body' diagram of the movable pulley

$$T_1=2T_2$$

$$\therefore\quad mg=2T_2+2T_2$$

or
$$T_2=\frac{mg}{4}$$

and
$$T_1=\frac{mg}{2}.$$

Here, $m=20+60=80$ kg. $\therefore\ T_1=40g=40$ kgf.

The force to be exerted by the man is 40 kgf. *Ans.*

27. Since the cannon stops after the shell is fired, the force down the plane, that is, $Mg\sin\alpha$ due to gravity must be balanced by the force exerted by the shell. By Newton's law the same force is exerted by the cannon on the shell which, in its turn, is equal to the rate of change of momentum of the shell.

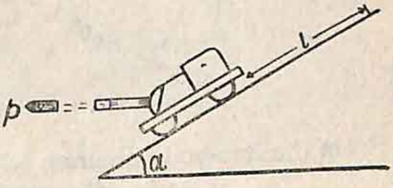


Fig. 2.66

Let us take downward along the plane as x-axis and line perpendicular to the plane as y-axis.

$$\vec{p_i} \text{ (initial momentum of the shell)} = mV\vec{i}$$

where V = velocity acquired by (cannon+shell) before firing

$$= \sqrt{2gl\sin\alpha}.$$

$$\vec{p_f} \text{ (final momentum of the shell)} = p\cos\alpha\,\vec{i} + p\sin\alpha\,\vec{j}.$$

$$\vec{F} \text{ (force exerted by the cannon on the shell)} = \frac{\vec{p_f} - \vec{p_i}}{\tau}$$

where τ is the duration of shot

or
$$\vec{F} = \frac{(p\cos\alpha - mV)\,\vec{i} + p\sin\alpha\,\vec{j}}{\tau}.$$

x-compoment of this force $= \dfrac{p\cos\alpha - mV}{\tau}$.

$$\therefore \quad \frac{p\cos\alpha - mV}{\tau} = Mg\sin\alpha$$

or
$$\tau = \frac{p\cos\alpha - m\sqrt{2gl\sin\alpha}}{Mg\sin\alpha}. \quad \textit{Ans.}$$

28. Since A tends to slip down, frictional forces act on it from both sides up the plane.

Let N be the reaction of the plane on A and N' be the mutual normal action-reaction between A and B.

We have from the free-body diagram of A,

$$N' + mg\cos\alpha = N$$

and
$$mg\sin\alpha = \mu(N+N').$$

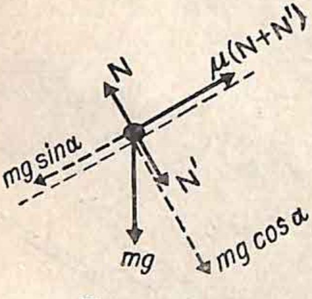


Fig. 2.67

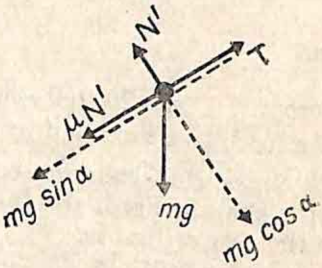


Fig. 2.68

From the free-body diagram of B,

$$N' = mg\cos\alpha;$$

$$mg\sin\alpha + \mu N' = T.$$

$$\therefore \quad 2mg\cos\alpha = N$$

and $$mg\sin\alpha=\mu(2mg\cos\alpha+mg\cos\alpha)$$
or $$\mu=\tfrac{1}{3}\tan\alpha;$$
$$\therefore\quad \mu=\tfrac{1}{3}\tan 37^\circ=\tfrac{1}{3}\times\tfrac{3}{4}=\cdot 25. \quad \textit{Ans.}$$

29. Let a be the acceleration of m down the incline relative to M. Then $a\cos\alpha$ and $a\sin\alpha$ are respectively leftward horizontal and vertical downward relative acceleration of m relative to M. Let a_x and a_y be absolute leftward and 'vertical downward' accelerations of m and b_x and b_y are absolute rightward and 'vertical downward' acceleration of M. Obviously $b_y=0$ as M has no vertical motion.

$a\cos\alpha=a_x-(-b_x)=a_x+b_x$ ($\because$ relative acceleration is vector difference of absolute accelerations)

$$a_x=a\cos\alpha-b_x$$

and $$a\sin\alpha=a_y-b_y=a_y-0=a_y$$
or $$a_y=a\sin\alpha.$$

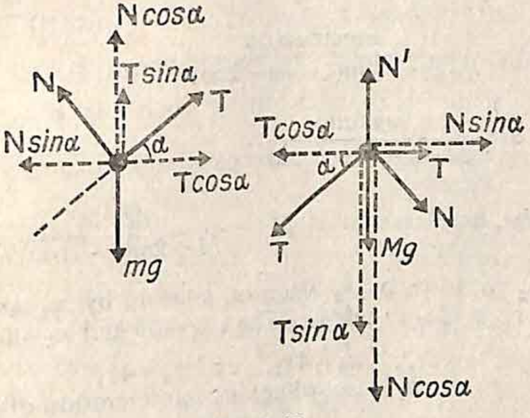


Fig. 2.69

From free-body diagram of m (resolving forces along the horizontal and vertical),

$$N\sin\alpha-T\cos\alpha=ma_x=m(a\cos\alpha-b_x)$$
$$mg-N\cos\alpha-T\sin\alpha=ma_y=ma\sin\alpha.$$

From free-body diagram of M (considering only horizontal motion),

$$T+N\sin\alpha-T\cos\alpha=Mb_x.$$

Since the string remains tensioned, the displacement of m down the plane must be equal to the rightward displacement of M, that is, *relative acceleration of m relative to M down the plane is equal to the absolute rightward acceleration of M.*

$$\therefore\quad a=b_x.$$

$$\therefore\quad N\sin\alpha-T\cos\alpha=ma\cos\alpha-ma \quad \text{(i)}$$
$$mg-N\cos\alpha-T\sin\alpha=ma\sin\alpha \quad \text{(ii)}$$
and $$T+N\sin\alpha-T\cos\alpha=Ma. \quad \text{(iii)}$$

We have to eliminate N and T and find a.

Eliminating N between (i) and (ii),

$$T = mg\sin\alpha + ma\cos\alpha - ma.$$

Eliminating N between (ii) and (iii),

$$T = \frac{mg\sin\alpha - ma\sin^2\alpha - Ma\cos\alpha}{1-\cos\alpha}.$$

$$\therefore \quad mg\sin\alpha + ma\cos\alpha - ma = \frac{mg\sin\alpha - ma\sin^2\alpha - Ma\cos\alpha}{1-\cos\alpha}$$

$$mg\sin\alpha - mg\cos\alpha - ma(1-\cos\alpha)^2 = mg\sin\alpha - ma\sin^2\alpha - Ma\cos\alpha$$

or $\quad ma(1-\cos\alpha)^2 = m$

or $\quad mg\sin\alpha\cos\alpha + ma(1-\cos\alpha)^2 = ma\sin^2\alpha + Ma\cos\alpha$

or $$a = \frac{mg\sin\alpha\cos\alpha}{M\cos\alpha + m(\sin^2\alpha - 1 + 2\cos\alpha - \cos^2\alpha)}$$

$$= \frac{mg\sin\alpha\cos\alpha}{M\cos\alpha + m(2\cos\alpha - 2\cos^2\alpha)}$$

or $$a = \frac{mg\sin\alpha}{M + 2m(1-\cos\alpha)}.$$

$\therefore \quad b_x$ (abs. acceleration of M) $= \dfrac{mg\sin\alpha}{M + 2m(1-\cos\alpha)}$. *Ans.*

30. Let m_2 go down by x_2 when m_1 goes up by x_1 and the axle-wheel system rotates through θ. Then $x_1 = a\theta$ and $x_2 = 4a\theta$.

$$\therefore \quad x_2 = 4x_1, \quad \text{or} \quad \ddot{x}_2 = 4\ddot{x}_1$$

$a_2 = 4a_1$ where a_2 = acceleration of m_2
a_1 = acceleration of m_1.

For downward motion of m_2,

$$m_2g - T_2 = m_2a_2 = 4m_2a_1. \qquad \text{.. (i)}$$

For upward motion of m_1,

$$T_1 - m_1g = m_1a_1. \qquad \text{.. (ii)}$$

Considering moment about the common axis,

$$T_2 \times 4a = T_1 \times a$$

or $$4T_2 = T_1. \qquad \text{.. (iii)}$$

We have to eliminate T_1 and T_2 to find a_1 between (i) and (ii) with the help of (iii).

or $\quad 4m_2g - 4T_2 = 16m_2a_1$

and $\quad 4m_2g - T_1 = 16m_2a_1$

$\quad\quad T_1 - m_1g = m_1a_1.$

Add, $\quad (4m_2 - m_1)g = (16m_2 + m_1)a_1$

or $$a_1 = \frac{4m_2 - m_1}{16m_2 + m_1} \times g. \quad \textit{Proved.}$$

31. Obviously the displacement of A down the incline is equal to the displacement of B to the left.

Hence A and B will have a common acceleration.

From free-body diagram of A (for motion down the plane only),

$$mg\sin\alpha - T = ma$$

and for horizontal motion of B,

$$T = ma.$$

$$\therefore \quad mg\sin\alpha = 2ma$$

or $$a = \tfrac{1}{2}g\sin\alpha = \tfrac{1}{2}g\sin 37^\circ = \tfrac{1}{2} \times 9{\cdot}8 \times \tfrac{3}{5}$$

(because the wedge is 3 : 4 : 5 right angled triangle)

or $$a = 2{\cdot}94 \text{ m/s}^2.$$

$$\therefore \quad 1 = 0 \times t + \tfrac{1}{2} \times 2\;94t^2 \quad (\text{from } s = ut + \tfrac{1}{2}ft^2).$$

$$t = \sqrt{\frac{2}{2{\cdot}94}} = {\cdot}825. \quad \textit{Ans.}$$

$$T \text{ (tension)} = 20 \times 2{\cdot}94 = 58{\cdot}8 \text{ newton.} \quad \textit{Ans.}$$

32. Let x be the instantaneous length on the left side.

Then $(l-x)$ is the length on the other side.

$$a \text{ (instantaneous acceleration)} = \frac{m_1 - m_2}{m_1 + m_2} g$$

$$= \frac{(l-x)m - xm}{(l-x)m + xm} \times g$$

or $$a = \frac{l-2x}{l} g = \left(1 - \frac{2x}{l}\right) g. \quad \textit{Ans.}$$

We have $$\frac{dv}{dt} = -\left(1 - \frac{2x}{l}\right) g$$

$\left(\because \; a = \dfrac{dv}{dt}\right.$ and the minus sign is put to account for the fact that when v increases x decreases)

or $$\frac{dv}{dx} \cdot \frac{dx}{dt} = \left(\frac{2x}{l} - 1\right) g$$

or $$v\,dv = \left(\frac{2x}{l} - 1\right) g\,dx \quad \left(\because \; \frac{dx}{dt} = v\right).$$

Integrating, $$\tfrac{1}{2}v^2 = \frac{gx^2}{l} - gx + c \text{ (a constant).}$$

Here $l = 10$ m, $v = 0$ when $x = 4$ m.

$$\therefore \quad 0 = \frac{g.4^2}{10} - g.4 + c$$

$$\therefore \quad \tfrac{1}{2}v^2 = \frac{g}{l}x^2 - gx + g4 - \frac{g}{10}4^2$$

or
$$\tfrac{1}{2}v^2 = g(4-x) - \frac{g}{10}(4^2 - x^2)$$

or
$$v^2 = 2g(4-x) - \frac{g}{5}(4^2 - x^2).$$

When $x=0,\ v^2 = 2g \times 4 - \frac{16}{5}g = \frac{24}{5}g.$

$$\therefore \quad v = \sqrt{\frac{24 \times 9{\cdot}8}{5}} = 6{\cdot}86 \text{ m/s.} \quad \textit{Ans.}$$

33. (a) Forward thrust in a rocket $= u_{rel}\,\dfrac{dM}{dt}$.

Here to overcome the weight 6000 kgf force is to be generated.

$$\therefore \quad 6000g = 1000\frac{dM}{dt}, \quad (\because \quad u_{rel} = 1 \text{ km/s} = 1000 \text{ m/s})$$

or
$$\frac{dM}{dt} = 6g = 6 \times 9{\cdot}8 = 58{\cdot}8 \text{ kg/s.} \quad \textit{Ans.}$$

(b) $F - 6000g = 6000 \times 19{\cdot}6$

$$F = 6000(9{\cdot}8 + 19{\cdot}6) = 6000 \times 3 \times 9{\cdot}8 = 1000\frac{dM}{dt}.$$

$$\therefore \quad \frac{dM}{dt} = 6 \times 3 \times 9{\cdot}8 = 176{\cdot}4 \text{ kg/s.} \quad \textit{Ans.}$$

34. Retarding force generated by sand drops

$$= \text{rate of change of momentum}$$

$$= v\frac{dM}{dt}.$$

Let F be the force needed to keep the belt moving with uniform velocity. Then

$$F - v\frac{dM}{dt} = 0, \quad (\because \quad \text{acceleration} = 0).$$

$$\therefore \quad F = v\frac{dM}{dt}. \quad \textit{Ans.}$$

Power = force × velocity $= v \times v\dfrac{dM}{dt} = v^2\dfrac{dM}{dt}.$ *Ans.*

35. We have, $F=-rv=ma$ where a is the retardation produced

or $$-rv=m\left(\frac{dv}{dt}\right), \quad \text{or} \quad \frac{dv}{v}=-\frac{r}{m}dt.$$

Integrating, $\log v=-\frac{r}{m}t+c$ (a constant).

At the moment $t=0$, $v=v_0$.

$$\therefore \quad \log v_0=c, \quad \therefore \quad \log v=-\frac{r}{m}t+\log v_0$$

or $$v=v_0e^{-\frac{r}{m}t}.$$

Obviously, $v=0$ when $t=\infty$. *Ans.*

(b) We have, $$-rv=m\frac{dv}{dt}$$

$$=mv\frac{dv}{dx}, \quad \left(\because \quad v=\frac{dx}{dt}\right)$$

or $$dv=-\frac{r}{m}dx.$$

Integrating, $v=-\frac{r}{m}x+c$ (a constant).

At the initial moment $v=v_0$ and $x=0$.

$$\therefore \quad v_0=c.$$

$$\therefore \quad v=v_0-\frac{r}{m}x. \quad \textit{Ans.}$$

At the final stage $v=0$.

$$\therefore \quad 0=v_0-\frac{r}{m}x_{total}, \quad \text{or} \quad x_{total}=\frac{mv_0}{r}. \quad \textit{Ans.}$$

36. Here $F=-kv^2$ where k is a constant

$=ma$ where a = retardation (a variable one) produced

$$=m\frac{dv}{dt}$$

or $\frac{dv}{dt}=-cv^2$ where $c=\frac{k}{m}$ (another constant)

or $$\frac{dv}{v^2}=-cdt.$$

Integrating, $$-\frac{1}{v}=-ct+\text{a constant.}$$

When $t=0$, $v=v_0$.

$$\therefore \quad -\frac{1}{v_0}=\text{the constant.}$$

$$\therefore \quad -\frac{1}{v}=-ct-\frac{1}{v_0}$$

or $$ct=\frac{1}{v}-\frac{1}{v_0}=\frac{v_0-v}{vv_0}, \quad \text{or} \quad t=\frac{v_0-v}{cvv_0} \quad .. \quad \text{(i)}$$

To find c we write the equation in terms of distance.

$$\frac{dv}{dt}=-cv^2$$

or $$v\frac{dv}{dx}=-cv^2, \quad \left(\because \quad \frac{dv}{dt}=\frac{dv}{dx}\times\frac{dx}{dt}=v\frac{dv}{dx}\right)$$

or $$\frac{dv}{v}=-cdx.$$

Integrating, $\log v=-cx+\text{a constant.}$

In the beginning $x=0$ and $v=v_0$.

$$\therefore \quad \log v_0=\text{the constant.}$$

$$\therefore \quad \log v=-cx+\log v_0.$$

When $x=h$, $v=v$ (given).

$$\therefore \quad \log v=-ch+\log v_0$$

or $$ch=\log\frac{v_0}{v}, \quad \text{or} \quad c=\frac{1}{h}\log\frac{v_0}{v}. \quad .. \quad \text{(ii)}$$

$$\therefore \quad t=\frac{h(v_0-v)}{v_0 v\log\frac{v_0}{v}}. \quad \textit{Ans.}$$

37. Consider free-body diagram of the body (for motion down the plane only).

$$mg\sin\alpha-kmg\cos\alpha=ma=m\frac{dv}{dt}, \quad \left(\because \quad a=\frac{dv}{dt}\right).$$

It is given that $k=as$.

$$\therefore \quad mg\sin\alpha-asmg\cos\alpha=m.\frac{vdv}{ds}, \quad \left(\because \quad \frac{dv}{dt}=\frac{dv}{ds}\cdot\frac{ds}{dt}=v\frac{dv}{ds}\right)$$

or $$g\sin\alpha - ag\cos\alpha \times s = v\frac{dv}{ds}$$

or $$g\sin\alpha ds - ag\cos\alpha \times s ds = v\,dv.$$

Integrating, $\frac{v^2}{2} = g\sin\alpha s - ag\cos\alpha\frac{s^2}{2} + \text{a constant.}$

In the beginning $s=0,\ v=0.$

$$\therefore\quad 0=0-0+\text{the constant.}$$

$$\therefore\quad \frac{v^2}{2} = g\sin\alpha \times s - \tfrac{1}{2}ag\cos\alpha s^2. \qquad .. \quad \text{(i)}$$

When it stops, $v=0$.

$$\therefore\quad g\sin\alpha \times s = \tfrac{1}{2}ag\cos\alpha s^2$$

or $$s = \frac{2}{a}\tan\alpha. \quad \textit{Ans.}$$

Differentiating (i), $v\frac{dv}{ds} = g\sin\alpha - ag\cos\alpha \times s.$

When the velocity is maximum, $\frac{dv}{ds}=0$

or $$s = \frac{1}{a}\tan\alpha.$$

$$\therefore\quad \tfrac{1}{2}v^2_{max} = g\sin\alpha \times \frac{1}{a}\tan\alpha - \tfrac{1}{2}ag\cos\alpha \times \frac{1}{a^2}\tan^2\alpha$$

$$v_{max} = \sqrt{\frac{g}{a}\sin\alpha\tan\alpha}. \quad \textit{Ans.}$$

38. Since the force varies with time, the body *will not start* moving immediately because unless the force attains the limiting frictional force, motion will not start. Let the motion start after t_0. Then

$$F = at_0 = kmg \text{ where } k = \text{coefficient of friction}$$

or $$t_0 = \frac{kmg}{a}. \qquad .. \quad \text{(i)}$$

So till $t \leqslant t_0$, $s=0$.

At the instant $t > t_0$, $F = a(t-t_0) = m\frac{dv}{dt}$

or $$dv = \frac{a}{m}(t-t_0)dt$$

Integrating, $v=\dfrac{a}{m}\tfrac{1}{2}(t-t_0)^2+$ a constant.

At the moment $t=t_0$, $v=0$;

$\therefore$ $0=$ the constant.

$$\therefore \quad v=\frac{a}{m}.\tfrac{1}{2}(t-t_0)^2, \quad \text{or} \quad \frac{ds}{dt}=\frac{a}{2m}(t-t_0)^2.$$

Integrating, $s=\dfrac{a}{6m}(t-t_0)^3+$ a constant.

At the moment $t=t_0$, $s=0$;

$\therefore$ the constant $=0$.

$$\therefore \quad s=\frac{a}{6m}(t-t_0)^3. \quad \textit{Ans.}$$

39. When m goes down by y, the wedge M moves to *the right* by $y\cot\alpha$. Thus if downward acceleration of m be a_y and rightward acceleration of M be b_x, then

$$a_y=\ddot{y} \quad \text{and} \quad b_x=\ddot{y}\cot\alpha=a_y\cot\alpha.$$

Considering free-body diagram of m (for vertical motion only),

$$mg-N\cos\alpha=ma_y \qquad .. \quad \text{(i)}.$$

Considering free-body diagram of M (for horizontal motion only),

$$N\sin\alpha=Mb_x=Ma_y\cot\alpha \qquad .. \quad \text{(ii)}.$$

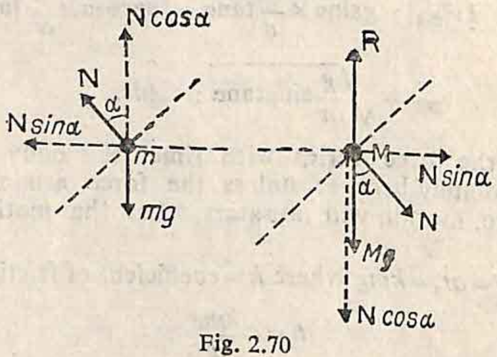


Fig. 2.70

Eliminating N between (i) and (ii),

$$\begin{aligned} -mg\sin\alpha-N\cos\alpha\sin\alpha&=ma_y\sin\alpha\\ N\sin\alpha\cos\alpha&=Ma_y\cot\alpha\cos\alpha; \end{aligned}$$

Add, $$mg\sin\alpha=a_y(m\sin\alpha+M\cot\alpha\cos\alpha)$$

$$a_y=\frac{mg\sin\alpha}{\sin\alpha(m+M\cot^2\alpha)}=\frac{mg}{m+M\cot^2\alpha}. \quad \textit{Ans.}$$

And $$b_x=a_y\cot\alpha=\frac{mg\cot\alpha}{m+M\cot^2\alpha}.\quad \textit{Ans.}$$

40. Considering free-body diagram of m_1 (resolving forces along the horizontal vertical),

$$T\cos\theta-\mu N=0$$
$$T\sin\theta+N=m_1g.$$

$$\therefore\quad T\cos\theta=\mu(m_1g-T\sin\theta),\quad \text{or}\quad T=\frac{\mu m_1g}{\cos\theta+\mu\sin\theta}.\quad \textit{Ans.}$$

Considering free-body diagram of m_2,

$$T-m_2g=0$$
$$T=m_2g.$$

$$\therefore\quad m_2g=\frac{\mu m_1g}{\cos\theta+\mu\sin\theta}$$

or $$\mu=\frac{m_2g\cos\theta}{m_1g-m_2g\sin\theta},\quad \text{or}\quad \mu=\frac{m_2\cos\theta}{m_1-m_2\sin\theta}.\quad \textit{Ans.}$$

41. (a) When A is projected to the right, B falls freely under gravity. The string is tensioned when A and B travel the same distance. Let t be time in which they cover the same distance. Then

ut (distance described by A) $=\frac{1}{2}gt^2$ (distance covered by B)

or $$t=\frac{2u}{g}.\quad \textit{Ans.}$$

(b) Let v be the initial common velocity. When the string is tensioned, the *two bodies experience the same impulsive force* but in the opposite direction.

Change in momentum of $A=2mv_f-2mu=2mv-2mu$.

Change in momentum of $B=mv-mu_{initial}$.

But $$u_{initial}=g.t=g.\frac{2u}{g}=2u.$$

$\therefore$ Change in momentum of $B=mv-m.2u$.

$\therefore$ By Newton's laws of motion,

$$(2mv-2mu)=-(mv-2mu)$$

or $$3mv=4mu,\quad \text{or}\quad v=\frac{4u}{3}.\quad \textit{Proved.}$$

(c) Let a be the common acceleration.

$$mg-T=ma$$

and $$T=2ma;$$

$$\therefore\quad a=\frac{g}{3}.$$

$$\therefore \quad v^2 - v_0^2 = 2l'.\frac{g}{3},$$

where l' is the distance covered by A or B with acceleration.

Now, $$l = l' + ut = l' + u.\frac{2u}{g}.$$

$$\therefore \quad l' = l - \frac{2u^2}{g}.$$

Here $$v_0 = \frac{4u}{3}.$$

$$\therefore \quad v^2 = \frac{16u^2}{9} + \frac{2g}{3}\left(l - \frac{2u^2}{g}\right) = \frac{16u^2}{9} + \frac{2gl}{3} - \frac{4u^2}{3} = \frac{4u^2+6gl}{9}.$$

$$\therefore \quad v = \frac{\sqrt{4u^2+6gl}}{3}. \quad Ans.$$

42. (a) When the difference of tensions on the two sides of an element of the string is equal to the limiting frictional force on the element, the string begins to slip.

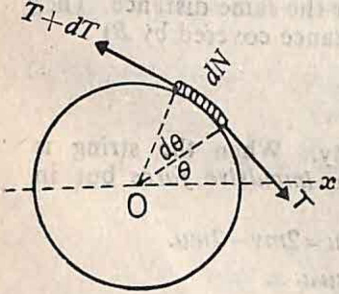


Fig. 2.71

$\therefore$ $dT = df_{lim} = \mu dN$ where dN is the normal reaction on the element.

Let $d\theta$ be the angular width of the element.

Then $$dN = T.\frac{d\theta}{2} + T.\frac{d\theta}{2} = Td\theta.$$

$$\therefore \quad dT = \mu Td\theta$$

or $$\frac{dT}{T} = \mu d\theta.$$

Integrating, $$\int_{T_1}^{T_2} \frac{dT}{T} = \mu\int_0^{\pi} d\theta, \quad \text{or} \quad \log\frac{T_2}{T_1} = \mu\pi.$$

But $m_2g - T_2 = 0$ (since m_2 has no acceleration)

or $$T_2 = m_2g$$

and $$T_1 = m_1g.$$

$$\therefore \quad \log\frac{m_2}{m_1} = \mu\pi$$

or $$\log n = \mu\pi, \quad \left(\because \quad \frac{m_2}{m_1} = n \text{ (given)}\right)$$

or $$\mu = \frac{1}{\pi}\log n. \quad \textit{Ans.}$$

(b) Now $m_2g - T_2 = m_2a$ where a = common acceleration
and $T_1 - m_1g = m_1a$.

$$\therefore \quad \frac{T_2}{T_1} = \frac{m_2(g-a)}{m_1(g+a)}.$$

Now $$\frac{T_2}{T_1} = n \quad \text{and} \quad \frac{m_2}{m_1} = N.$$

$$\therefore \quad n = \frac{N(g-a)}{g+a}$$

or $$a = \frac{N-n}{N+n}g. \quad \textit{Ans.}$$

43. (a) Since $F = at(\tau - t)$, the force acts for. τ seconds.

Now $$F = \frac{dp}{dt}.$$

$$\therefore \quad \frac{dp}{dt} = at(\tau - t)$$

or $$dp = a\tau t\, dt - at^2 dt.$$

Integrating, $p = \frac{1}{2}a\tau\, t^2 - \frac{1}{3}at^3 + a$ constant.
At the moment $t = 0, p = 0$;
$\therefore$ the constant $= 0$.
$\therefore$ $p = \frac{1}{2}a\tau\, t^2 - \frac{1}{3}at^3$.

For $t = \tau$, $p = \frac{1}{2}a\tau.\tau^2 - \frac{1}{3}a\tau^3 = \frac{a\tau^3}{6}$. *Ans.*

(b) $$v = \frac{p}{m} = \frac{a\tau}{2m}.t^2 - \frac{a}{3m}t^3$$

or $$ds = \frac{a\tau}{2m}t^2dt - \frac{a}{3m}t^3dt, \quad \left(\because \quad v = \frac{ds}{dt}\right).$$

Integrating, $s = \frac{a\tau}{6m}\, t^3 - \frac{a}{12m}t^4 +$ a constant.

At the moment $t = 0, s = 0$;
$\therefore$ the constant $= 0$.

$$\therefore \quad s=\frac{a\tau}{6m}\,t^3-\frac{a}{12m}t^4.$$

For $t=\tau$, $s=\dfrac{a\tau^4}{6m}-\dfrac{a\tau^4}{12m}=\dfrac{a\tau^4}{12m}$. *Ans.*

Solutions to Test Questions

1. (i) A man runs for a certain distance after alighting from a moving train to maintain the same relative velocity of all parts of his body relative to the train so that he may not fall forward due to inertia for rest of the lower part of his body and inertia for motion of the upper of his body.

(ii) Because of greater distance and hence greater time in which his motion is stopped, he experiences less force.

(iii) An athlete runs over a certain distance before taking a long jump to gain inertia for motion so that he may take longer jump.

(iv) Due to inertia for rest ripe mangoes get detached from their weak stem.

(v) Suppose $2l$ is the length of the string and $2a$ is the distance between the points on the upper edge of the frame where the ends of the string are attached. Then inclination of each part of the string with the horizontal is $\cos^{-1}\dfrac{a}{l}$. If T is the tension of the string, then for equilibrium of the picture we have,

$$W=2T\sin\theta=2T\sqrt{1-\frac{a^2}{l^2}}.$$

Thus it is seen shorter length means greater tension. So the picture is likely to fall if the length is too short.

2. (i) True; (ii) True; (iii) False; (iv) True; (v) False, momentum is conserved only when the external force on the system is zero.

3. (i) (c); (ii) (b); (iii) (b); (iv) (b); (v) (d).

4. (a) $F=\dfrac{\text{Change in momentum}}{\text{Time}}$

$$=\frac{150\times10^{-3}\times20-(-150\times10^{-3}\times12)}{\cdot01}$$

or $$F=480 \text{ newtons.} \quad \textit{Ans.}$$

(b) Let x be the maximum length that can hang freely. Mass of the hanging part is xm where m is the mass per unit length. The driving force on the rest is 'xmg'. The limiting frictional force on the rest of the chain is $\mu(l-x)mg$.

$$\therefore \quad xmg = \mu(l-x)mg.$$

$$\therefore \quad x = \frac{\mu l}{\mu+1}. \quad \textit{Ans.}$$

5. (a) The 'free-body' diagrams of M and m are shown.

Here f_{lim} = limiting frictional force; R = interaction between M and m; N = reaction of the ground.

Considering only horizontal motion of M.

$$F - R = ma$$

or $\quad R = F - Ma. \quad \ldots$ (i)

Considering both vertical and horizontal motion of m,

$f_{lim} = mg$, or $kR = mg$

$(\because \; f_{lim} = kR)$

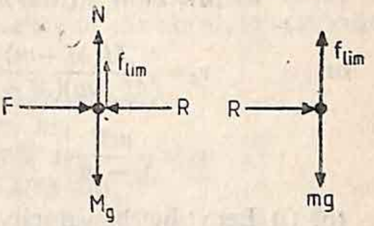


Fig. 2.72

or $\quad R = \dfrac{mg}{k} \quad \ldots$ (ii)

and $\quad R = ma.$

$$\therefore \quad ma = \frac{mg}{k}, \quad \text{or} \quad a = \frac{g}{k}. \quad \ldots \text{(iii)}$$

Substituting (ii) and (iii) in (i),

$$\frac{mg}{k} = F - M\frac{g}{k}, \quad \text{or} \quad F = (M+m)\frac{g}{k}. \quad \textit{Ans.}$$

(b) Considering free-body diagram (only for horizontal motion) we have

$$5 - R_1 = 2a$$
$$R_1 - R_2 = 3a$$
$$R_2 = 5a$$

Add, $\quad 5 = 10a \quad$ or $\quad a = \frac{1}{2}.$

$\therefore \quad R_1 = 5 - 1 = 4$ newton. *Ans*

$R_2 = 2{\cdot}5$ newton. *Ans.*

6. (a) Let v_1 and v_2 be the velocities of the boats 1 and 2 respectively. Boat 1 is moving to the right and boat 2 is moving to the left.

The sack from 2 brings *leftward momentum* to 1 by mv_2 and the sack from 1 carries away 'rightward momentum' by mv_1.

$\therefore$ Net momentum (rightward) of boat 1 after exchange of sacks

$$= (M+m)v_1 - mv_2 - mv_1 = 0$$

(as it stops after exchange)

or $\quad Mv_1 = mv_2. \quad \ldots$ (i).

Similarly, net leftward momentnm of boat 2

$$= (M+m)v_2 - mv_1 - mv_2$$
$$= (M+m)V \quad \text{(given)}$$

or $v_2-\frac{m(v_1+v_2)}{M+m}=V$, or $\frac{M}{M+m}v_2-\frac{M}{M+m}v_1=V$. .. (ii)

Putting (i) in (ii),

$$\frac{M}{M+m}v_2-\frac{m^2}{(M+m)M}v_2=V$$

or $$v_2=\frac{M(M+m)V}{(M+m)(M-m)}=\frac{MV}{M-m}.\quad Ans.$$

$$v_1=\frac{mV}{M-m}.\quad Ans.$$

(b) (i) Let v' be the velocity of the car to the left after both men jump off simultaneously to the right. Let u_{ab} be the absolute velocity of the men to the right after they jump off. Then

$u=u_{ab}-(-v')$ ($\because$ relative velocity is the vector difference of absolute velocities).

Since the system is free from external forces in the horizontal, momentum is conserved in this direction.

Considering conservation of 'momentum to the left' we have

$$0=Mv'-2mu_{ab}=Mv'-2m(u-v')$$

or $$v'=\frac{2mu}{M+2m}.\quad Ans.$$

(ii) When the first man jumps off.

$$0=(M+m)v'-mu_{ab}$$

or $$(M+m)v'=m(u-v')$$

or $$v'=\frac{mu}{M+2m}.$$

Let v'' be the velocity of the car to the left when the second man jumps off. Then

$$u=u_{ab}-(-v'')$$

or $$u_{ab}=u-v''.$$

Considering conservation of momentum to the left,

$$(M+m)v'=Mv''-mu_{ab}$$
$$=Mv''-m(n-v'')$$
$$=(M+m)v''-mu$$

or $$v''-v'=\frac{mu}{M+m}$$

or $$v''=v'+\frac{mu}{M+m}=\frac{mu}{M+2m}+\frac{mu}{M+m}$$

or $$v''=\left(\frac{mu}{M+m}+\frac{mu}{M+2m}\right),\quad Ans.$$

Obviously velocity in the second case is greater than the velocit in the first case.

7. Let a be the absolute upward acceleration of the monkey anc a' be the downward absolute acceleration of the rope. Obviously a is also the rightward acceleration of M. Then

$b=a-(-a')$ ($\because$ relative acceleration is the vecto difference of the absolute accelerations

or $\quad b=a+a'$.

Considering upward motion of the monkey,

$$T-mg=ma. \quad .. \quad \text{(i}$$

Considering rightward motion of M,

$$T=Ma'=M(b-a). \quad .. \quad \text{(ii}$$

Eliminating a between (i) and (ii) we get T in terms of b.

$$T=\frac{mM}{m+M}\cdot(g+b).$$

(i) $b=0$.

$$\therefore \quad T=\frac{mM}{m+M}g. \quad \textit{Ans.}$$

(ii) $T=\frac{mM}{m+M}(g+b)$. *Ans.*

(iii) Now b is to be replaced by $-b$.

$$\therefore \quad T=\frac{mM}{m+M}(g-b). \quad \textit{Ans.}$$

8. Let a be the relative acceleration of m down the plane relative to M. Let a_x and a_y be the absolute rightward and vertical downward accelerations of m and b_x be the leftward acceleration of M.

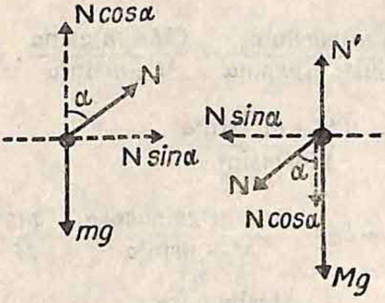


Fig. 2.73

Then $\quad a\cos\alpha=a_x-(-b_x)$ ($\because$ relative acceleration is the vecto difference of absolute accelerations

$a\cos\alpha=a_x+b_x$

and $\quad \sin\alpha = a_y - 0 \quad$ (as M has no vertical acceleration)

$$= a_y.$$

Considering motion of m along the vertical and horizontal direction,

$$mg - N\cos\alpha = ma_y \quad \text{or} \quad mg - N\cos\alpha = ma\sin\alpha \qquad \text{.. (i)}$$

$$N\sin\alpha = ma_x \quad \text{or} \quad N\sin\alpha = m(a\cos\alpha - b_x) \qquad \text{.. (ii)}$$

Considering free-body diagram of M (only for horizontal motion),

$$N\sin\alpha = Mb_x \qquad \text{.. (iii)}$$

Putting the value of N from (iii) in (i) and (ii) we have,

$$mg - \frac{Mb_x}{\sin\alpha}.\cos\alpha = ma\sin\alpha$$

or

$$mg\sin\alpha - Mb_x\cos\alpha = ma\sin^2\alpha \qquad \text{.. (iv)}$$

$$Mb_x = ma\cos\alpha - mb_x$$

or

$$(M+m)b_x = ma\cos\alpha \qquad \text{.. (v).}$$

Dividing (iv) by (v) to eliminate a and find b_x,

$$\frac{mg\sin\alpha - Mb_x\cos\alpha}{(M+m)b_x} = \frac{ma\sin^2\alpha}{ma\cos\alpha}$$

or

$$mg\sin\alpha\cos\alpha - Mb_x\cos^2\alpha = (M+m)b_x\sin^2\alpha$$

or

$$b_x = \frac{mg\sin\alpha\cos\alpha}{M + m\sin^2\alpha}.$$

$$N = \frac{M}{\sin\alpha} \times b_x = \frac{Mmg\cos\alpha}{M + m\sin^2\alpha}.$$

Putting the value of N in (i) we have the value of a,

$$mg - \frac{Mmg\cos^2\alpha}{M + m\sin^2\alpha} = ma\sin\alpha$$

or

$$a = \frac{(M+m)mg\sin^2\alpha}{(M + m\sin^2\alpha)m\sin\alpha} = \frac{(M+m)g\sin\alpha}{M + m\sin^2\alpha}.$$

$$\therefore \quad a_y = a\sin\alpha = \frac{(M+m)g\sin^2\alpha}{M + m\sin^2\alpha}$$

and

$$a_x = a\cos\alpha - b_x = \frac{(M+m)g\sin\alpha\cos\alpha}{M + m\sin^2\alpha} - \frac{mg\sin\alpha\cos\alpha}{M + m\sin^2\alpha}$$

$$= \frac{Mg\sin\alpha\cos\alpha}{M + m\sin^2\alpha}.$$

Thus, acceleration of $M = \dfrac{mg\sin\alpha\cos\alpha}{M + m\sin^2\alpha}$. *Ans.*

Horizontal acceleration of $m=\dfrac{Mg\sin\alpha\cos\alpha}{M+m\sin^2\alpha}$. *Ans.*

Vertical downward acceleration of $m=\dfrac{(M+m)g\sin^2\alpha}{M+m\sin^2\alpha}$. *Ans.*

Mutual reactions $=\dfrac{Mmg\cos\alpha}{M+m\sin^2\alpha}$. *Ans.*

9. a (acceleration) $=\dfrac{m_1-m_2}{m_1+m_2}g$.

Here $m_1+m_2=M$ (a constant)

or $$a=\frac{m_1-m_2}{M}g \qquad \text{.. (i)}$$

T (tension) $=\dfrac{2m_1m_2}{m_1+m_2}g$, or $m_1m_2=\dfrac{TM}{2g}$ .. (ii).

Squaring (i), $$a^2=\frac{g^2}{M^2}(m_1-m_2)^2$$

or $$a^2=\frac{g^2}{M^2}[(m_1+m_2)^2-4m_1m_2]$$

$$=\frac{g^2}{M^2}\left[M^2-4.\frac{TM}{2g}\right]$$

$$a^2=\left(g^2-\frac{2T}{M}g\right).$$

Obviously acceleration is the least when T is maximum. It is given that $T_{max}=$ breaking weight $=\frac{15}{32}Mg$.

$$\therefore\ a^2{}_{min}=\left(g^2-\frac{2.\frac{15}{32}Mg}{M}g\right)=\frac{g^2}{16}$$

or $$a_{min}=\frac{g}{4}.\quad \textit{Proved.}$$

$$\therefore\ \frac{g}{4}=\frac{m_1-m_2}{M}g$$

or $$m_1-m_2=\frac{M}{4}.$$

But $$m_1+m_2=M$$

$$\therefore\ m_1=\tfrac{5}{8}M \text{ and } m_2=\tfrac{3}{8}M.$$

Thus the least value of the greater mass $=\frac{5}{8}$ M. Prov .

10. The same frictional force is effective on A and B. This force produces retardation on A and acceleration on B till they acquire a common velocity.

$F=ma=Ma'$ where $a=$ absolute retardation of m

$a'=$ absolute acceleration of M.

Relative retardation of $m=a-(-a')=a+a'$.

Initial relative velocity $=v_0$.

Final relative velocity $=0$.

$$\therefore \quad v_0^2=2(a+a')s$$

where $s=$ distance covered by m relative to M

or $$v_0^2=2\left(\frac{F}{m}+\frac{F}{M}\right)s=\frac{2F(m+M)}{mM}s$$

or $$s=\frac{mMv_0^2}{2F(m+M)}. \quad \textit{Ans.}$$

LESSON 3

Topics : **Moment of a Force and Torque**
Centre of Mass and Centre of Gravity
Machines
Equilibrium of Bodies

3.1 Moment of a Force and Torque or Couple

Moment of a Force. The turning effect of a force about a point, called 'the moment centre', is called 'moment of a force' and is given by $\tau = Fr\sin\theta$ where F is the magnitude of the force, r is the mod of the position vector of the point of application of the force (p) from the 'moment centre' (C) and θ is the angle between F and $\mathbf{r}$. But $r\sin\theta$ represents the perpendicular distance 'p' of the 'moment centre' from the line of action of the force. Therefore, the moment of a force about a point is also given by $\tau = F.p$.

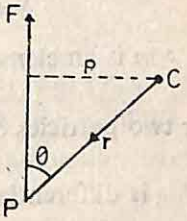


Fig. 3.1

The moment of a force is positive if it turns or tends to turn the body anticlockwise and negative in the clockwise direction. However, this may be considered the other way round as well.

Couple or Torque. A pair of equal and anti-parallel forces acting on the same body is called a couple. The moment or torque of the couple is measured by the moment of the two forces about any point (P, Q, R). This is a constant, irrespective of the point chosen, and is equal to the product of either force and the perpendicular distance between the forces. Thus

$$\tau = F.d$$

where F is the magnitude of either force of the couple and d is the distance between them, called 'arm of the couple'. The effect of applying a couple on a body is to rotate the body.

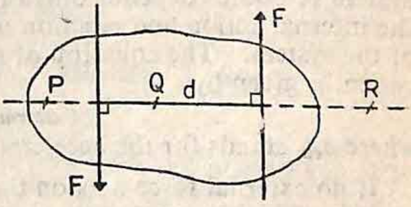


Fig. 3.2

3.2 Centre of Mass and Centre of Gravity

Centre of Mass. The centre of mass of a rigid body or a system of bodies is that point where the whole mass may be supposed to be concentrated; the body or the system of bodies may be considered as a single particle so far as the action of a system of parallel forces acting on the constituent particles of the body or system of bodies is concerned. The concept of the centre of mass of a system of bodies is very helpful in statics and dynamics. It reduces the problem of a rigid body or a system of bodies to a 'particle problem'.

If M is the mass of the body and $m_1, m_2, m_3, \ldots$ are the elementary masses at positions $\vec{r}_1, \vec{r}_2, \vec{r}_3 \ldots$ with respect to an arbitrary frame of reference, then the position vector of the centre of mass is given by

$$M\vec{r} = m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \ldots$$

The x and y coordinates of the centre of mass is given by

$$x_{cm} = \frac{\Sigma[\triangle m]x}{\Sigma\triangle m}, \quad y_{cm} = \frac{\Sigma[\triangle m]y}{\Sigma\triangle m}$$

where $\triangle m$ is an elementary mass of coordinates (x, y).

For two particles of masses m_1 and m_2, $x_{cm} = \dfrac{m_1x_1 + m_2x_2}{m_1 + m_2}$.

If this is differentiated with respect to time, we get

$$\dot{x}_{cm} = \frac{m_1\dot{x}_1 + m_2\dot{x}_2}{m_1 + m_2}.$$

Thus the velocity of the centre of mass ($\dot{x}_{cm}$) is obtained from the individual velocities of the two particles ($\dot{x}_1$ and $\dot{x}_2$). Care should be taken to allot proper signs to $\dot{x}_1$ and $\dot{x}_2$.

Motion of Centre of Mass. The motion of centre of mass of a body or system of bodies depends only on the external forces and never on the internal action and reaction of the constituent particle or bodies of the system. The equation of motion of the centre of mass of a system is given by

$$F_{external} = Ma_{cm}$$

where a_{cm} stands for the acceleration of the centre of mass.

If no external force acts on the system, then $F_{external} = 0$ and so $a_{cm} = 0$. This means that there is no change in velocity of the centre of mass. So if the body or system of bodies were at rest in the beginning, then its centre of mass would remain at rest though its constituent parts might have changed their positions.

Centre of Gravity. The centre of gravity is that point where the whole mass of a body or system of bodies may be supposed to be concentrated so far as the gravity (that is, force of attraction due to the earth) on the particles of the body or system is concerned. For bodies which are small compared to the earth, the centre of mass and centre of gravity are identical points. Hence the centre of gravity of such bodies are given by the above expressions for centre of mass of a system of particles.

3.3 Machines

Machines. A machine is a device by which force applied at some point of it (called the load) is overcome by means of another force applied at some other point (called the 'effort'). The load and effort may be different in direction or magnitude or both.

Real Mechanical Advantage.

$$\text{Real mechanical advantage of a machine } (A)=\frac{\text{Load } (W)}{\text{Effort } (E)}.$$

$$\text{Efficiency of a machine } (\eta)=\frac{\text{Useful work}}{\text{Total work}}$$

$$\text{i.e.,}\quad \frac{\text{Work done on load (output)}}{\text{Work done by effort (input)}}.$$

Work done by a force = force × displacement along the direction of the force.

Velocity Ratio or Ideal Mechanical Advantage.

$$\text{Velocity ratio } (r)=\frac{\text{Velocity of effort}}{\text{Velocity of load}}$$

$$\text{or}\quad \frac{\text{displacement of effort } (x)}{\text{displacement of load } (y)}.$$

Ideal mechanical advantage (A_{ideal}) is the advantage had there been no friction.

$\therefore\ W\times y=E\times x$ under ideal conditions.

$\therefore\ A_{ideal}=\dfrac{W}{E}=\dfrac{x}{y}=$ velocity ratio. This can be calculated from simple geometrical considerations.

In any machine, Mechanical advantage = Efficiency × Velocity ratio

or $A_{real}=\eta\times r$, or $A_{real}=\eta\times A_{ideal}$.

Lever, axle and wheel, capstan, jack etc. are practical machines. Among these machines, some work on the principle that 'moment of load about any point = moment of effort about that point', provided they are ideal. Examples are lever, axle and wheel, capstan etc. Some machines are practical applications of 'screw and nut'. Examples are jack, vice etc. In all ideal machines, work done by effort = work done on load.

3.4 Equilibrium of Bodies

A rigid body is said to be in equilibrium when its linear acceleration and angular acceleration are zero.

Conditions for equilibrium of a body are :

$$\Sigma \vec{F} = 0 \quad \text{and} \quad \Sigma \vec{\tau} = 0.$$

Here, $\Sigma \vec{F}$ and $\Sigma \vec{\tau}$ represent the resultants of the external forces and external torques acting on the body. Thus, for a body to be in equilibrium :

(a) the sum of the external forces acting on it in any direction must be zero, and

(b) the sum of the torques acting on it must be zero.

While applying these conditions, the forces and torques must be given their proper signs (+ve or −ve).

Principle of stability of equilibrium :

A system is in stable equilibrium when its potential energy is a minimum. This is a universal principle.

From the dynamics point of view a system in gravitational field is in stable equilibrium when the vertical line through its centre of gravity passes through the base of the body.

CRITICAL QUESTIONS

1. The effect of applying a couple on a body is to rotate the body. True of false ?

Ans. True.

2. The moment of a force about a point is a scalar. True or false ?

Ans. False. The moment of a force about a point is an axial vector.
$$\vec{\tau} = \vec{r} \times \vec{F}.$$

2. (a) Give an example in which the body is not in equilibrium, even though the resultant of all forces acting on it is zero.

Ans. A body rotating about an axis under the action of a couple.

3. If a body is not in translational equilibrium, will the torque about any point be zero if the torque about some particular point is zero ?

Ans. No. When a body is not in translational equilibrium, there is some resultant force on the body. The moment of the forces about a point on the line of action of the resultant is zero but moment about any point off the line of action of the resultant cannot be zero.

4. Which is more likely to break, a hammock stretched tightly between two supports or one that sags quite a bit ? Prove your answer.

Ans. A hammock is a hanging bed in ships. For equilibrium of hammock of weight W we have (see figure) $W=2T\cos\theta$.

$$T=\frac{W}{2\cos\theta}. \quad \therefore \quad T\propto\sec\theta.$$

Hence the greater the tension, the greater is the angle θ. So the hammock is likely to break when it is tightly stretched between supports.

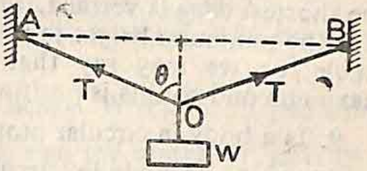


Fig. 3.3

5. A ladder is at rest with its upper end against a wall and the lower end on the ground. Is it more likely to slip when a man stands on it at the bottom or at the top ? Explain. **(C. E. E. 1975)**

Ans. When the man is at the top the moment of his weight about the lower end is greater than that when he is at the bottom. So the ladder is likely to slip when he is at the top.

6. How can a spring balance be used to weigh objects beyond the maximum reading of the balance ? **(N. D. A. 1977)**

Ans. Take a rigid rod and place its one end on a raised support (see figure). Suspend the body to be weighed at the centre of the rod. Hook up the other end by the spring balance and raise this end till the rod becomes horizontal. This arrangement just doubles the range of the spring balance. If it is a compression type spring balance, interchange the positions of the spring balance and the body to be weighed.

$$W\times l/2=R\times l$$

(neglecting the weight of the rod)

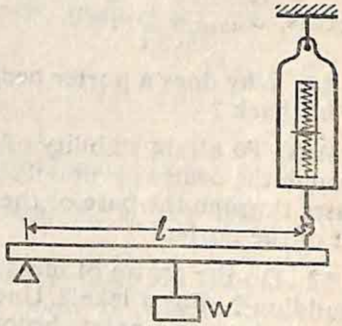


Fig. 3.4

where R is the reading of the spring balance

or $$W=2R.$$

7. When a tree is cut down, the cut is made on the side facing the direction in which tree is required to fall. Explain. (NDA 1977)

Ans. When the cut is deep and is almost near the other end, the weight of the tree will have moment about that end. Due to this moment of the weight of the tree it will fall in the direction in which the tree is cut.

8. A uniform block, in the shape of a rectangular parallelopiped of sides in the ratio 1 : 2 : 3, lies on a horizontal surface. In what position would its equilibrium be most stable ?

Ans. When it lies on the face containing the edges 2 and 3 and the shortest edge is vertical. The reason is that the centre of gravity is at the minimum height in this position and so this position is most stable, or we may say that in this position area of the base is maximum and so this is position of most stable equilibrium.

9. Is a body in circular motion in equilibrium ?

Ans. No. A body in circular motion undergoes a centripetal acceleration $\vec{a}$ directed towards the centre of the circle. Since $\vec{a} \neq 0$ the body is not in equilibrium.

10. A wheel rotating at constant angular velocity ω about a fixed axis is in mechanical equilibrium because no net external force or torque acts on it. However, the particles that make up the wheel undergo a centripetal acceleration $\vec{a}$ directed toward the axis. Since $\vec{a} \neq 0$ how can the wheel be said to be in equilibrium ?

Ans. The acceleration of the wheel (which is the vector sum of the accelerations of the particles that make up the wheel) is zero because the particles on the two sides of the axis of rotation possess equal and opposite accelerations and so their vector sum is zero. That is, $\vec{a}_{wheel} = \sum\limits_i \vec{a}_i = 0$.

11. Why does a porter bend forward while carrying a sack of rice on his back ?

Ans. To attain stability of equilibrium so that the vertical line through the centre of gravity of the system (porter+sack of rice) passes through the base of the system. that is, the space between the feet of the porter.

12. Do the centre of mass and the centre of gravity coincide for a building ? For a lake ? Under what conditions does the difference between these two points become insignificant ?

Ans. Yes. No. When the size of the body is small in comparison to the radius of the earth.

13. Is it necessary that there be any mass at the centre of mass of a system ?

Ans. No. Example, a ring.

14. Does the centre of mass of a solid necessarily lie within the body ? If not, give an example.

Ans No. An L-shaped rod.

15. If only an external force can change the state of motion of the centre of mass of a body, how does it happen that the internal force of the brakes can bring a car to rest ?

Ans. In fact the internal force does not bring the car to rest. The internal force of the brake on the wheel locks the wheel. Now an enormous frictional force arises between the wheel and the ground. This force is external to the system and this external force brings the car to rest.

16. The bottom of a ship is made heavy. Why ? **(NDA 1981)**

Ans. The bottom is made heavy so that the centre of gravity of the ship occupies the lowest position and stability of equilibrium is ensured.

17. In a single pulley there is no mechanical advantage. Even then it is a machine. Justify.

Ans. In a single pulley such as the one used to draw water from a well there is no mechanical advantage but there is directional advantage. To draw water one has to apply vertically downward force. So it is a machine.

18. It is said that 'what is gained in power is lost in speed'. Is it true of false ?

Ans. It is true. In any machine

Mechanical advantage $=$ efficiency $\times$ velocity ratio.
Thus, mechanical advantage $\propto$ velocity ratio.

$$\text{But velocity ratio} = \frac{\text{velocity of effort}}{\text{velocity of load}}.$$

$$\therefore \text{ Mechanical advantage} \propto \frac{1}{\text{velocity of load}}.$$

Thus greater the mechanical advantage, less is the speed of the load.

19. In a stationary railway compartment there are 100 passengers. They start fisting each other on some point of difference in their opinions. Will the centre of mass of the compartment change its position ? Will the centre of mass of the system (compartment$+$100 passengers) change its position ? Explain.

Ans. The position of the centre of mass of the compartment will change position because the passengers are external bodies for the compartment. But the centre of mass of the system will not change position as there is no external force on the system.

20. A system is in stable equilibrium when its potential energy is minimum. Is it true or false ?

Ans. True.

21. Why is it easier for two men to turn a log about its centre than to move it translationally ?

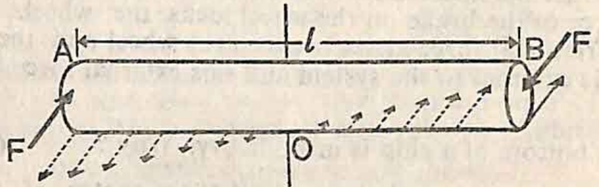


Fig. 3.5

Ans. To turn the log, the maximum moment of the forces applied by the two men should be greater than the moment of the limiting frictional forces about the centre of the log. If l is the length of the log then the maximum moment of the forces applied by the men is $F.\, l$, where F is the force applied by each man. The forces of friction are distributed along the log (see figure). The mean force of friction on each side of the centre is $\mu\frac{Mg}{2}$ where M is the mass of the log and μ is the coefficient of friction.

The two frictional forces on the two sides of the centre form a couple of arm $\frac{l}{2}$. Hence the moment of the limiting frictional forces is $\mu\frac{Mg}{2}\times\frac{l}{2}$. To just turn the log we must have

$$F.l=\mu\frac{Mg}{2}\times\frac{l}{2} \quad \text{or} \quad F=\frac{1}{4}\mu Mg.$$

To move the log translationally, the resultant force applied by the men must overcome the limiting frictional force. Thus just to move the log we must have

$$2F=\mu Mg \quad \text{or} \quad F=\frac{\mu Mg}{2}.$$

This is how it is easier to turn a log than to move it translationally.

22. A rectangular block rests on a horizontal table. A horizontal force is applied to move it on the table at a height h above the table. Does the line of action of the normal force N exerted by the table on block depend on h ?

Ans. Yes. As the height h is increased, the normal reaction shifts to the corner about which the block tends to tip.

23. A homogenous rectangular brick lies on an inclined plane. What half of the brick (lower or upper) exerts a greater thrust on it ?

Ans. Let us consider the two halves as two separate bolies, each of weight half the weight of the brick. Over the surface of separation the upper half exerts a force R parallel to the plane downward and the lower half exerts the same force on the upper half. Considering free-body diagrams of upper and lower half we have

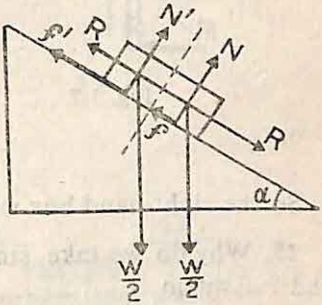


Fig. 3.6

$$\frac{W}{2}\sin\alpha+R=f$$

$$N=\frac{W}{2}\cos\alpha$$

and $$\frac{W}{2}\sin\alpha-R=f'.$$

$$\therefore\ N'=\frac{W}{2}\cos\alpha.$$

Note. Here the frictional forces are *not* μN and $\mu N'$ because the brick *is* not in limiting equilibrium. Here frictional forces are f and f' on the lower and upper half.

Obviously $f>f'$ and $N=N'$.

Now the thrust is the vector sum of the normal reaction and the tangential reaction, that is, frictional force.

$$\therefore\ T=\sqrt{N^2+f^2} \text{ and } T'=\sqrt{N'^2+f'^2}=\sqrt{N^2+f'^2}.$$

$$\because\ f>f',$$

$$\therefore\ T>T'.$$

So the lower half exerts a greater thrust.

24. A rod not reaching the rough floor is inserted between two identical boxes (see figure). A horizontal force F is applied to the upper end of the rod. Which of the boxes will move first ? Explain.

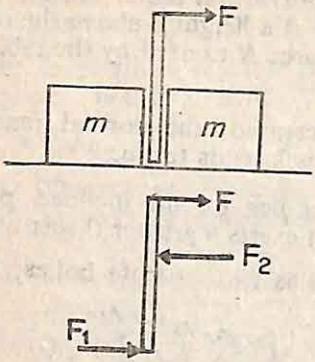


Fig. 3.7

Ans. Consider the free-body diagram of the rod and apply conditions for its equilibrium. Let F_1 be the force exerted by the rod on the left box. Then F_1 is the force exerted by the box on the rod to the right. Let F_2 be the force exerted by the rod on the right-hand box. Then F_2 is the force exerted by it on the rod to the left. For equilibrium of the rod we have

$$F_1+F=F_2$$

or

$$F=F_2-F_1$$

$$\therefore \quad F_2>F_1.$$

So the right-hand box will move first.

25. Why do we take small steps while walking on a slippery road ? Explain.

Ans. When a man walks the line joining his centre of gravity C to the toe 'O' of his foot is inclined to the vertical by certain angle α. The value of this angle depends on the size of the step the man takes. The smaller the step, the less is the value of α. Let F be the reaction of the ground on the toe of the man directed towards the centre. The horizontal component of this force cannot be greater than the maximum value of frictional force, otherwise he will slip. Therefore, for equilibrium

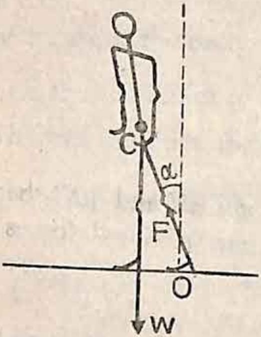


Fig. 3.8

$$F\sin\alpha=f_{friction} \text{ and } F\cos\alpha=mg.$$

But

$$f_{friction}<f_{lim}.$$

$$\therefore \quad F\sin\alpha<\mu F\cos\alpha$$

or

$$\mu>\tan\alpha.$$

When μ is small, α is also small. This is why small steps are taken while walking on slippery road.

26. Can a uniform ladder be prevented from falling by a rope tied to its centre at one end and at the other end to a peg fixed at the intersection of the ground and the wall if both are smooth ?

Ans. No, because for equilibrium moment of all the forces about any point must be zero. Consider the point of intersection of the normals to the wall and the ground as the 'moment centre'. The moments of N_1, N_2 and T about O are zero as they pass through it but the moment of W remains unbalanced and hence the ladder cnnnot be prevented from falling however great the tension may be.

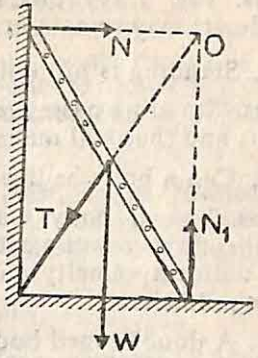


Fig. 3.9

27. Two identical weights are connected by a spring. At the initial moment the spring is so compressed that the first is tightly pressed against a wall (see figure) and the second weight is clamped by a stop '*S*'. How will the weights move if the stop is removed ? Explain without calculations.

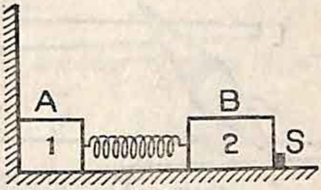


Fig. 3.10

Ans. When the stop is removed, the weight 2 will move to the right with acceleration due to the action of the spring to the right and the weight 1 will remain pressed against the wall. When the spring regains its normal length, the second weight will gain some velocity. The velocity of centre of mass at this instant is

$$V_{em}=\frac{m_1 0+m_2 v_2}{m_1+m_2}=\frac{m_2 v_2}{m_1+m_3}.$$

So long the system was not free from external force along the horizontal as the wall exerted force on 1. At the moment the spring regains its original length, the system becomes free from external force and so the velocity of the centre of mass of the system will remain fixed at v_{cm}, but 1 and 2 will be respectively accelerated and retarded. So they come closer with such velocities that the velocity of the centre of mass remains constant. Again they move away from each other till they are farthest. Thus, the system as a whole will move away from the wall with constant velocity, but the weights alternately converge to each other and diverge from each other.

28. Some heavy boxes are to be loaded along with some empty boxes on a cart. Which boxes should be put on the cart first and why ?

Ans. The heavy boxes should be loaded first so that cg of the loaded cart may remain at the lowest position.

29. Standing is prohibited in a double decker bus. Explain.

Ans. Standing passengers in the upper deck will raise the cg of the bus and that will make it less stable.

30. Can a body be in equilibrium while in motion ? Explain.

Ans. Yes, a body may be in equilibrium while it is in motion. Equilibrium means 'no acceleration—linear and angular. If a body is in uniform velocity along a straight line, the body will be said to be in equilibrium.

31. A doubl-coned body P is placed on two parallel horizontal rails AB and CD which have slight upward slope from the left to right as shown in the figure. When the body is released, it rolls down. But if the ends A and C are drawn closer so as to make some angle between AB and CD, it rolls up instead of going down. Explain why.

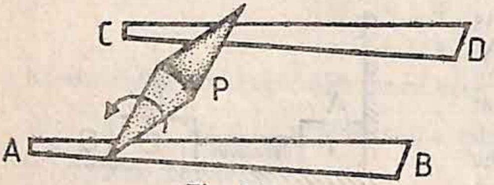


Fig. 3.11

Ans. The principle involved is 'that a body tends to occupy that position where it has minimum potential energy'. When the rails are parallel, the cg of the body is lowered as it goes down and so it does so. When the rails are inclined as indicated in the figure, the cg is lowered as it goes up and so it goes up.

32. Why cannot we rise from a chair without bending a little forward ?

Ans. Because the weight forms a torque about the feet and that makes us unable to rise from the chair. When we bend forward and bring cg of the body above the feet, the couple disappears and so now it becomes possible to rise from the chair.

OBJECTIVE QUESTIONS

1. If a machine is lubricated with oil

(a) the mechanical advantage of the machine increases,

(b) the efficiency of the machine increases,

(c) both its mechanical advantage and efficiency increase,

(d) its efficiency increases, but the mechanical advantage decreases.

(IIT 1980)

Ans. (c). In a machine where there is friction, the effort has to overcome the load and frictional force. So in the presence of friction the machine can overcome less load. This accounts for its low mechanical advantage. When it is lubricated, frictional force vanishes and so the same effort can now overcome greater load. Thus its mechanical advantage is increased. Some energy is wasted in overcoming frictional force. This accounts for low efficiency of a machine When lubricated, no energy is wasted and so efficiency of the machine also increases simultaneously.

2. Two particles A and B initially at rest, move towards each other under a mutual force of attraction. At the instant when the speed of A is V and the speed of B is $2V$, the speed of the centre of mass of the system is

(a) zero, (b) V, (c) $1{\cdot}5V$, (d) $3V$. (IIT 1982)

Ans. (a) Since the system is free from external force, the acceleration of the centre of mass is zero. The initial velocity of the centre of mass is zero and so the velocity of the centre of mass of system in subsequent instants is also zero.

3. A block of mass m rests on a wedge of mass M, which, in turn, rests on a horizontal table (see figure). All surfaces are frictionless. If the block starts from rest,

(a) the centre of mass of the system will change its position,

(b) the centre of mass of the system will change position along the vertical but not along the horizontal,

(c) the centre of mass of the system will change position along the horizontal but not along the vertical,

(d) nothing can be said about the position of the centre of mass.

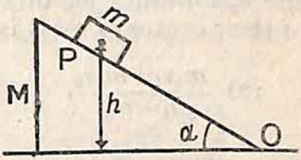


Fig. 3.12

Ans. Since the system is free from external force along the horizontal there will be no change in the position of the centre of mass along the horizontal but the system is not free from external forces along the vertical as gravitational pull is acting on the system, there will be change in position of the centre of mass along the vertical. So the answer is (b).

4. Two particles P and Q are initially at rest at 1 m apart. P has a mass of 1 kg and Q a mass of 3 kg. Due to mutual attraction P and Q move towards each other. They collide at distance

(a) 0·25 m from P, (b) 0·25 m from Q, (c) 0·5 m from Q,

(d) 0·5 m from P.

Ans. The centre of mass is at distance $=\dfrac{1\times 1}{1+4}= {\cdot}25$ m from Q.

Since the system is free from external force, the centre of mass of the system will remain stationary. So they collide at their centre of mass which is at a distance 0·25 m from Q. So the answer is (b).

5. Two particles P and Q of mass 1 kg and 3 kg respectively start moving towards each other under their mutual attraction with velocities 30 ms^{-1} and 6 ms^{-1}. The velocity of their centre of mass is (a) 36 ms^{-1}, (b) 18 ms^{-1}, (c) 3 ms^{-1}, (d) none of these.

Ans. We have $Mx=m_1x_1+m_2x_2$ or $M\dot{x}=m_1\dot{x}_1+m_2\dot{x}_2$.

$$\therefore \quad \dot{x}=\frac{m_1\dot{x}_1+m_2\dot{x}_2}{m_1+m_2} \quad (\because \quad M=m_1+m_2)$$

$$=\frac{1\times30+3(-6)}{1+3}=3\ \text{ms}^{-1}.$$

This is the initial velocity of the centre of mass. Since the system is free from external force, the acceleration of the centre of mass is zero. So the velocity of the centre of mass remains unchanged in the subsequent instants. Therefore, the answer is (c).

6. If two particles of masses m_1 and m_2 move with velocities v_1 and v_2 towards each other on a smooth horizontal table, the velocity of their centre of mass is

(a) $\dfrac{m_1v_1+m_2v_2}{m_1+m_2}$, (b) $\dfrac{m_1v_1-m_2v_2}{m_1+m_2}$, (c) $\dfrac{m_1v_2+m_2v_1}{m_1+m_2}$,

(d) $\dfrac{m_1v_2-m_2v_1}{m_1+m_2}$.

Ans. (b).

7. An inclined plane forms an angle α to the horizontal. The maximum mechanical advantnge of it as machine is (a) cosecα, (b) tanα, (c) cotα, (d) sinα.

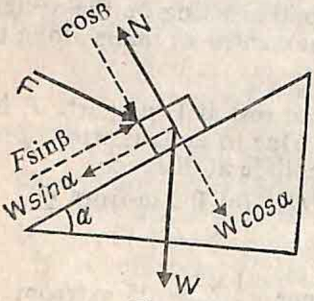


Fig. 3.13

Ans. Let a load W be raised up the plane by a force F inclined at β with the normal to the plane. Resolving along and perpendicular to the plane we have

$$W\sin\alpha=F\sin\beta$$
and $$N=W\cos\alpha+F\cos\beta.$$

$$\text{Mech. advantage}=\frac{W}{F}$$

$$=\sin\beta\ \text{cosec}\alpha.$$

$\therefore$ Maximum mech. advantage $=\text{cosec}\alpha\times(\sin\beta)_{max}$.

But $(\sin\beta)_{max}=1$.

$\therefore$ Max. mech. advantage $=\text{cosec}\alpha$.

So the answer is (a).

8. If α is the angle of inclination of a plane to the horizontal and λ is the angle of friction between the plane and the body to be raised, the maximum mechanical advantage is

(a) $\operatorname{cosec}\alpha$, (b) $\sin\alpha$,
(c) $\operatorname{cosec}(\alpha+\lambda)$, (d) $\operatorname{cosec}(\alpha-\lambda)$.

Ans. Imposing an additional force μN down the plane in the above figure (3.13) we have $W\sin\alpha = F\sin\beta - \mu N$

and $$N = W\cos\alpha + F\cos\beta.$$

$$\therefore \quad W\sin\alpha = F\sin\beta - \mu W\cos\alpha - \mu F\cos\beta.$$

$$\therefore \quad \text{A (mechanical advantage)} = \frac{W}{E} = \frac{\sin\beta - \mu\cos\beta}{\sin\alpha + \mu\cos\alpha} = \frac{\sin(\beta-\lambda)}{\sin(\alpha+\lambda)}$$

(putting $x = \tan\lambda$)

$\therefore \quad A_{max} = \operatorname{cosec}(\alpha+\lambda)$. So the answer is (c).

9. In a lever if the effort arm is doubled

(a) the mechanical advantage is doubled and efficiency is also doubled, (b) mech. adv. is doubled but not the efficiency, (c) efficiency is doubled but not the mech. adv., (d) both remain unchanged.

Ans. Let E be the effort and W be the load and l_1 and l_2 be the initial effort and load arm of the lever.

Then A_i (initial mech. adv.)

$$= \frac{W}{E} = \frac{l_1}{l_2}$$

$(\because \quad W \times l_2 = E \times l_1)$

A_f (final mech. adv.)

$$= \frac{2l_1}{l_2} = 2A_i.$$

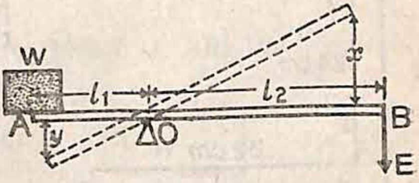


Fig. 3.14

$$\eta_i \text{ (initial efficiency)} = \frac{\text{Output}}{\text{Input}} = \frac{Wy}{Ex} = \frac{l_1}{l_2} \times \frac{l_2}{l_1}$$

$\left(\because \quad \frac{W}{E} = \frac{l_1}{l_2} \text{ and } \frac{y}{x} = \frac{l_2}{l_1} \right.$ from similar triangles OAA' and OBB')

$$= 1.$$

$$\eta_f \text{ (final efficiency)} = \frac{2l_1}{l_2} \times \frac{l_2}{2l_1} = 1.$$

$$\therefore \quad \eta_i = \eta_f.$$

So the answer is (b).

10. If in the axle and wheel the radii of the axle and wheel are respectively 5 cm and 25 cm, the mech. adv. (A) and efficiency (η) (neglect friction) are (a) $A=5, \eta=1$, (b) $A=5, \eta=5$, (c) $A=1. \eta=5$, (d) $A=1, \eta=1$.

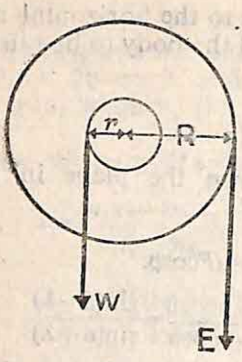


Fig. 3.15

Ans. Considering moment about the common axis,

$$W \times r = E \times R.$$

$$\therefore \quad A = \frac{W}{E} = \frac{R}{r} = \frac{25 \text{ cm}}{5 \text{ cm}} = 5.$$

$$\eta = \frac{\text{Output}}{\text{Input}} = \frac{W \times 2\pi R}{E \times 2\pi R} = \frac{W}{E} \times \frac{E}{W} = 1.$$

So the answer is (a).

11. The position coordinates $(\bar{x}, \bar{y})$ of the centre of gravity of the L-shaped plate with reference to the origin at lower left-hand corner are

(a) $\bar{x} = 11$ cm, $\bar{y} = 9$ cm, (b) $\bar{x} = 7$ cm, $\bar{y} = 9$ cm,

(c) $\bar{x} = 3$ cm, $\bar{y} = 10$ cm, (d) $\bar{x} = 14$ cm, $\bar{y} = 8$ cm.

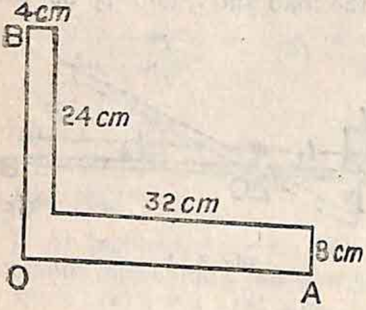


Fig. 3.16

Ans. Since it is a plate, mass is proportional to the area. Hence the total mass is

$$(4 \times 24 + 8 \times 36) = 384.$$

$$\therefore \quad 384 \times \bar{x} = (96 \times 2 + 288 \times 18)$$

$$\bar{x} = 14 \text{ cm}$$

$$384 \times \bar{y} = (96 \times 20 + 288 \times 4)$$

$$\bar{y} = 8 \text{ cm}.$$

12. The position coordinates $(\bar{x}, \bar{y})$ of the centre of mass of the T-shaped plate with reference to the frame shown in the figure are

(a) $\bar{x} = 0$, $\bar{y} = 8$, (b) $\bar{y} = 0$, $\bar{x} = \frac{20}{3}$,

(c) $\bar{y} = 0$, $\bar{x} = \frac{20}{3}$ cm, (d) $\bar{x} = 0$, $\bar{y} = \frac{20}{3}$ cm.

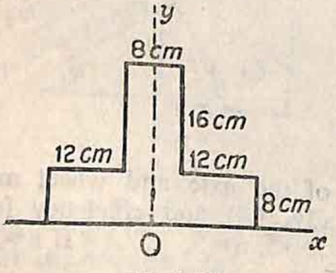


Fig. 3.17

Ans. We have

$$(256 + 128)\bar{y} = 256 \times 4 + 128 \times 16$$

$$\bar{y} = \frac{3072}{384} = 8 \text{ cm}.$$

Since the body is symmetrical about O_y, $\bar{x} = 0$.

So the answer is (a).

13. The ideal mechanical advantage of the device (shown in the figure) is (a) 2, (b) 3, (c) 4, (d) 5.

Ans. Consider free-body diagram of W.

$$W = T_1 + T_2.$$

From the free-body diagram of the movable pulley

$$2T_2 = T_1.$$

From the free-body diagram of an element of the string at the end where the effort is applied

$$T_2 = F.$$

$$\therefore \quad W = 2F + F = 3F$$

$$\therefore \quad A_{ideal} = \frac{W}{F} = 3.$$

So the answer is (b).

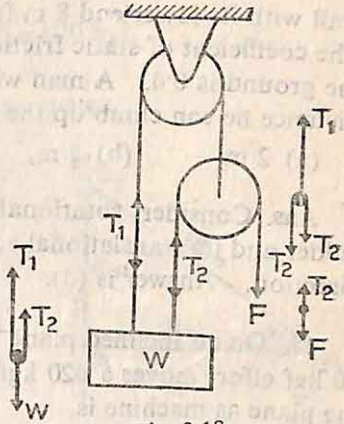


Fig. 3.18

14. A screw jack moves upward 6 mm when turned through one revolution. The ideal mechanical advantage of it when a 50 cm long bar is used to turn is

(a) $\frac{500}{6}$, (b) $\frac{500\pi}{3}$, (c) 300π, (d) 16π.

Ans. Work done on load in one revolution $= W \times p$.

Work done by effort in one revolution $= E \times 2\pi l$.

In ideal conditions

$$W \times p = E \times 2\pi l$$

$$\therefore \quad A_{ideal} = \frac{W}{E} = \frac{2\pi l}{p}$$

$$= \frac{2\pi \times 50 \text{ cm}}{6 \text{ mm}} = \frac{500\pi}{3}.$$

So the answer is (b).

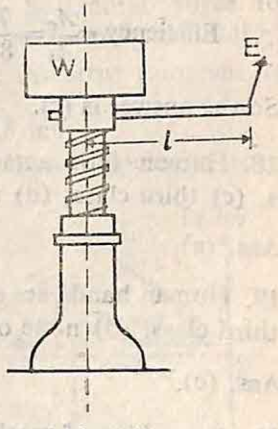


Fig. 3.19

15. A door 4 m high and 2 m wide is hung from hinges 3 m and $\frac{1}{2}$ m from the top and the bottom of the door. The door weighs 42 kg, its centre of gravity is at its centre, and each hinge carries half the weight of the door. If H_1 and H_2 are the horizontal components of the force exerted on the door at the hinges then

(a) $H_1 = 10$ kgf, $H_2 = 10$ kgf, (b) $H_1 = 1$ kgf, $H_2 = 3$ kgf,
(c) $H_1 = H_2 = 14$ kgf, (d) $H_1 = 21$ kgf and $H_2 = 12$ kgf.

Ans. Consider 'moment about the lower hinge'. Answer is (c).

16. A uniform ladder 10 m long rests against a frictionless vertical wall with its lower end 8 m from the wall. The ladder weighs 40 kg. The coefficient of static friction between the foot of the ladder and the ground is 0·4. A man weighing 60 kg starts up the ladder. The distance he can climb up the ladder before it starts to slip is

(a) 2 m, (b) $\frac{3}{2}$ m, (c) 5 m, (d) $\frac{5}{3}$ m.

Ans. Consider rotational equilibrium about the foot of the ladder and its translational equilibrium along the vertical horizontal direction. Answer is (d).

17. On an inclined plane 10 m long and 1·25 m at the raised end, a 60 kgf effort moves a 420 kgf load up the plane. The efficiency of the plane as machine is

(a) 90%, (b) 70%, (c) 87·5%, (d) 75%.

Ans. A_i (ideal MA) $= \text{cosec}\theta = \dfrac{10 \text{ m}}{1{\cdot}25 \text{ m}} = 8.$

A_r (real MA) $= \frac{420}{60} = 7.$

$\therefore$ Efficiency $= \dfrac{A_r}{A_i} = \dfrac{7}{8} = {\cdot}875 = 87{\cdot}5\%.$

So the answer is (c).

18. Human foot acts as a lever of (a) first class, (b) second class, (c) third class, (d) none of these.

Ans. (a).

19. Human hands act as levers of (a) first class, (b) second class, (c) third class, (d) none of these.

Ans. (c).

20. A machine of mechanical advantage 5 needs a 72 kgf effort to raise a load. If the efficiency of the machine is reduced to 60% due to friction, the effort needed is

(a) 72 kgf, (b) 100 kgf, (c) 120 kgf, (d) 150 kgf.

Ans. (c). $A_i = 5 = \dfrac{W}{72}$ or $W = 360$ kg.

Efficiency $=\frac{A_r}{A_i}$, $\therefore$ $\cdot 6=\frac{A_r}{5}$, or $A_r=3$.

$$\therefore\ 3=\frac{360}{E}, \text{ or } E=120 \text{ kgf}.$$

NUMERICAL QUESTIONS

1. The strut *AB* in the Fig. weighs 20 kg and its centre of gravity is at its centre. A load of 60 kg is suspended at the far end. Find the tension in the cable *AC* and the horizontal and vertical components of the force exerted on the strut at the wall.

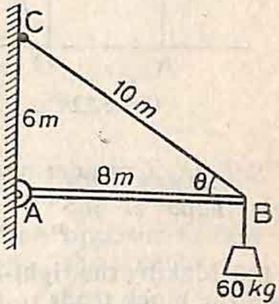


Fig. 3.20

Sol. Considering 'translational equilibrium' of the strut we have $60+20=T\sin\theta+V$ (equating vertical downward forces to upward forces)

and $H=T\cos\theta$ (equating rightward horizontal force to the leftward horizontal force).

Considering 'rotational equilibrium' of the strut (moment centre *A*) we have

$$20\times 4+60\times 8=T.8\sin\theta$$

(equating clockwise moment to anticlockwise moment).

There are three equations and three unknowns *T*, *H* and *V*.

From the simple geometry $\sin\theta=\frac{6}{10}=\frac{3}{5}$ and $\cos\theta=\frac{8}{10}=\frac{4}{5}$

$\therefore\ 80=T\times\frac{3}{5}+V$

or $400=3T+5V$ .. (i)

$H=T\frac{4}{5}$

or $5H=4T$ .. (ii)

$560=T.8.\frac{3}{5}$

or $350=3T$ .. (iii)

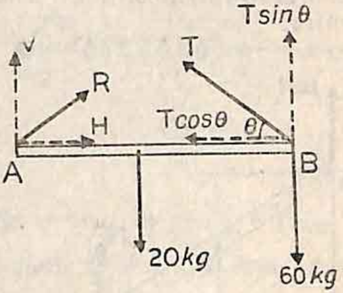


Fig. 3.21

Hence, $\left.\begin{array}{l}T=116{\cdot}67 \text{ kgf}\\ H=93{\cdot}33 \text{ kgf}\\ V=14 \text{ kgf}\end{array}\right\}$. *Ans.*

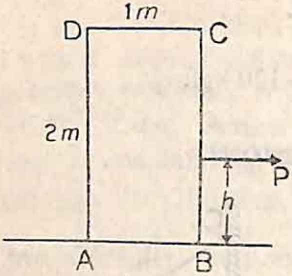


Fig. 3.22

2. A rectangular block 1 m wide and 2 m high is dragged to the right along a level road at constant speed by a horizontal force P, as shown in the figure. The coefficient of sliding friction is 0·4, the block weighs 50 kg and its centre of gravity is at its centre. (a) Find the force required. (b) Find the value of h at which the block just starts to tip. (c) Find the line of action of the normal force N exerted on the block by the surface, if the height h is 1 m.

Sol. (a) Considering horizontal and vertical equilibrium we have

$$50g-N=0 \quad \text{and} \quad P-\mu N=0.$$

$$\therefore \quad P=\mu N=\cdot 4\times 50\times 9{\cdot}8=196 \text{ N.} \quad \textit{Ans.}$$

(b) Making the right-hand corner B the 'moment centre' we have when the block tends to tip about B

$P.h=50g\times\frac{1}{2}$ (the normal reaction is now at B and so its moment about B is zero)

or $196h=50\times 9{\cdot}8\times\frac{1}{2}$, $h=1{\cdot}25$ m. *Ans.*

(c) Considering 'moment about' B

$$196\times 1+N\times d=50g\times\tfrac{1}{2}$$

or $$196+50g\times d=25g, \text{ or } d=\cdot 1$$

$\therefore$ Distance from the centre $=\cdot 5$ m $-\cdot 1$ m $=\cdot 4$ m to the right. *Ans.*

3. A ladder of weight W rests in limiting equilibrium against a vertical well. The coefficient of friction between the wall and the upper end of the ladder is μ and that between its lower end and the ground is μ'. Calculate the inclination of the ladder to the horizontal in this position. The centre of gravity of the ladder is at its centre.

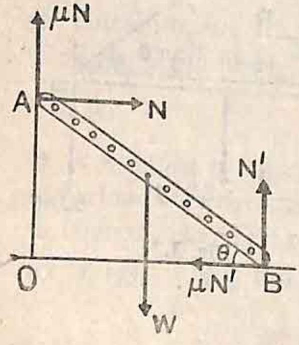


Fig. 3.23

Sol. For translational equilibrium we have $\mu N+N'=W$

and $\mu' N'=N$

$$\therefore \quad (\mu\mu'+1)N'=W \quad \text{.. (i)}$$

Considering 'rotational equilibrium' about the upper end ($2l$ = length of the ladder)

$$W\times l\cos\theta+\mu' N'\times 2l\sin\theta = N'\times 2l\cos\theta$$

or $$(1+\mu\mu')N'+\mu' N'\times 2l\sin\theta = N'\times 2l\cos\theta$$

$$(1+\mu\mu')\cos\theta+2\mu'\sin\theta=2\cos\theta$$

or $$\tan\theta=\frac{1-\mu\mu'}{2\mu'}\cdot \quad \textit{Ans.}$$

4. A pulley fixed to the ceiling carries a thread with bodies of masses m_1 and m_2 attached to its ends. The masses of the pulley and the thread are negligible, friction is absent. Find the acceleration a_{cm} of the centre of mass of the system.

Sol. Let x_{cm} be the distance of the centre of mass from the centre of the pulley at any instant and x_1 and x_2 are the distances of m_1 and m_2 from the same point.

$$x_{cm}=\frac{m_1x_1+m_2x_2}{m_1+m_2}.$$

$$\therefore\quad \ddot{x}_{cm}=\frac{m_1}{m_1+m_2}\ddot{x}_1+\frac{m_3}{m_1+m_2}\ddot{x}_2.$$

or $a_{cm}=\dfrac{m_1}{m_1+m_2}a_1+\dfrac{m_2}{m_1+m_2}(-a_1)$ ($\because$ the acceleration of one is equal and opposite to that of the other)

$$=\left(\frac{m_1-m_2}{m_1+m_2}\right)a_1.$$

Now $$a_1=\left(\frac{m_1-m_2}{m_1+m_2}\right)\times g.$$

$$\therefore\quad a_{cm}=\left(\frac{m_1-m_2}{m_1+m_2}\right)^2\times g.\quad \textit{Ans.}$$

5. Two small discs of masses m_1 and m_2 are connected by a weightless spring rest on a smooth horizontal plane. The discs are set in motion with initial velocities v_1 and v_2 whose directions are mutually perpendicular and lie in the same horizontal plane. Find the total energy E of this system with reference to the frame fixed to the centre of mass.

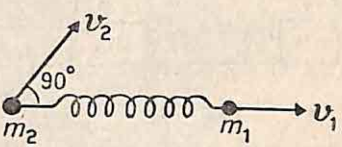


Fig. 3.24

Sol. Let $\vec{v}_{cm}$ be the velocity of the centre of mass. Then the velocity of m_1 relative to the centre of mass is $\vec{v}_1-\vec{v}_{cm}$ and that of m_2 is $\vec{v}_2-\vec{v}_{cm}$.

$$\therefore\quad E=\tfrac{1}{2}m_1\,|\,\vec{v}_1-\vec{v}_{cm}\,|^2+\tfrac{1}{2}m_2\,|\,\vec{v}_2-\vec{v}_{cm}\,|^2.$$

The velocity of the centre of mass is given by

$$(m_1+m_2)\vec{v}_{cm}=m_1\,\vec{v}_1+m_2\,\vec{v}_2.$$

or
$$v_{cm}=\frac{m_1\vec{v}_1+m_2\vec{v}_2}{m_1+m_2}.$$

$$\therefore\quad \vec{v}_1-\vec{v}_{cm}=\frac{m_2(\vec{v}_1-\vec{v}_2)}{m_1+m_2}.$$

$$\therefore\quad |\vec{v}_1-\vec{v}_{cm}|=\frac{m_2}{m_1+m_2}|\vec{v}_1-\vec{v}_2|$$

$$=\frac{m_2}{m_1+m_2}(v_1^2+v_2)^{\frac{1}{2}}$$ ($\because$ v_1 and v_2 are at right angles).

Similarly, $$|\vec{v}_2-\vec{v}_{cm}|=\frac{m_1}{m_1+m_2}(v_2^2+v_1^2)^{\frac{1}{2}}.$$

$$\therefore\quad E=\tfrac{1}{2}m_1\left(\frac{m_2}{m_1+m_2}\right)^2.(v_1^2+v_2^2)+\tfrac{1}{2}m_2\left(\frac{m_1}{m_1+m_2}\right)^2(v_1^2+v_2^2)$$

$$=\tfrac{1}{2}.\frac{m_1m_2}{(m_1+m_2)^2}(v_1^2+v_2^2)(m_2+m_1)$$

$$=\tfrac{1}{2}\mu(v_1^2+v_2^2)\quad\text{where}\quad \mu=\frac{m_1m_2}{m_1+m_2}$$

called the reduced mass. *Ans.*

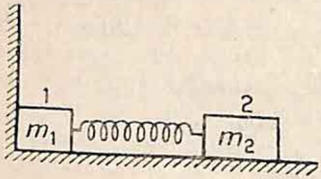


Fig. 3.25

6 Two bars of masses m_1 and m_2 connected by a weightless spring of force constant k rest on a smooth horizontal plane. Bar 2 is shifted a small distance x to the left and then released. Find the velocity of the centre or mass of the system after the bar 1 breaks off the wall.

Sol. When the bar 1 breaks off the wall its velocity is zero and let v_2 be the velocity of the bar 2. Then velocity of the centre of mass is given by

$$(m_1+m_2)v_{cm}=m_1\times 0+m_2v_2,\quad\text{or}\quad v_{cm}=\frac{m_2v_2}{m_1+m_2}.$$

From conservation of energy we have

$$\tfrac{1}{2}kx^2=\tfrac{1}{2}m_2v_2^2,\quad\text{or}\quad v_2=\sqrt{\frac{k}{m_2}}x.$$

$$\therefore\quad v_{cm}=\frac{m_2}{m_1+m_2}\sqrt{\frac{k}{m_2}}x=\frac{x}{m_1+m_2}\sqrt{km_2}$$

Since the system is free from external force, the acceleration of its centre of mass is zero. So the velocity of the centre of mass is always $v_{cm}=\dfrac{x}{m_1+m_2}\sqrt{km_2}$. *Ans.*

7. A raft of mass M with a man of mass m aboard stays motionless on the surface of a lake. The man moves a distance l relative to the raft with velocity $u(t)$ and then stops. Assuming the water resistance to be negligible, find

(a) the displacement of the raft relative to the shore;

(b) the horizontal component of the force with which the man acted on the raft during the motion.

Sol. Let x_1 and x_2 be the initial distances of the centres of mass of the raft and the man from the shore and x_1' and x_2' be the final distances of their centres of mass from the shore.

Then $x_{cm}=\dfrac{Mx_1+mx_2}{M+m}$ and $x'_{cm}=\dfrac{Mx_1'+mx_2'}{M+m}$.

Since the system is free from external force

$$x_{cm}=x'_{cm}.$$

$$\therefore \quad Mx_1+mx_2=Mx_1'+mx_2'$$

or $$M(x_1-x_1')=m(x_2'-x_2) \qquad \text{.. (i).}$$

Now absolute displacement (that is, displacement relative to the shore) of the raft $=x_2'-x_2$

and absolute displacement of the man (that is, displacement relative to the shore) $=x_1'-x_1=l'$ (say).

$\therefore$ Displacement of the man relative to the raft

$$=(x_1'-x_1)-(x_2'-x_2)=l \text{ (given)}$$

or $$l'-(x_2'-x_2)=l, \quad \text{or} \quad x_2'-x_2=l'-l \qquad \text{.. (ii).}$$

$\therefore$ From (i), $$M(-l')=m(l'-l)$$

or $$l'=\frac{ml}{M+m}. \quad \textit{Ans.}$$

(b) The force exerted by the man on the raft is effective on the raft.

$$\therefore \quad F=M\times\frac{dv}{dt}$$

here v is the absolute velocity of the raft. If the instantaneous absolute displacement of the raft is l', then

$$v=\frac{dl'}{dt}=\frac{m}{M+m}\cdot\frac{dl}{dt}=\frac{m}{M+m}.u, \quad \left(\because \frac{dl}{dt}=u \text{ given}\right).$$

$$\therefore \quad \frac{dv}{dt}=\frac{m}{M+m}\cdot\frac{du}{dt}.$$

$$\therefore \quad F=\frac{mM}{M+m}\cdot\frac{du}{dt}. \quad \textit{Ans.}$$

8. The front wall of drawer in a cabinet is provided with two symmetrical handles. The distance between the handles is l and the length of the drawer is a. The coefficient of friction between the drawer and the cabinet is μ. Show that the drawer can be pulled out by applying a force perpendicular to the drawer to one of its handles if $\mu<\frac{a}{l}$.

Sol. Let us denote the force applied to one handle by F. The force will turn the drawer and induce normal reactions and frictional forces as shown. For tanslational equilibrium we have

$$N_1=N_2=N \text{ (say)}$$

and condition for pulling out the drawer is $F\geqslant\mu(N_1+N_2)$

or $F\geqslant 2\mu N$.

For rotational equilibrium about A

$$F(l+b)=Na+\mu N(l+2b)$$

where b is the distance of each handle from the nearer edge.

For rotational equilibrium about C

$$Fb+Na=\mu N(l+2b).$$

Subtracting we have

$$Fl=2Na$$

$$F=\frac{2Na}{l}.$$

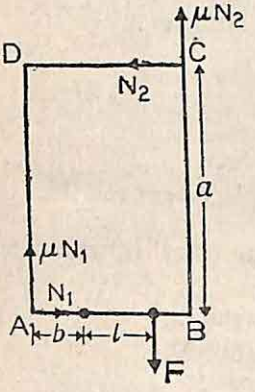


Fig. 3.26

$\therefore$ Condition for pull-out is

$$\frac{2Na}{l}\geqslant 2\mu N$$

or

$$\mu\leqslant\frac{a}{l}. \quad \textit{Proved.}$$

9. A weightless inextensible string is wound around a cylinder of mass m. With what minimum force and at what angle α to the horizon should the string be pulled for the rotating cylinder to remain in place ? The coefficient of friction between the cylinder and the floor is μ.

Sol. The cylinder is in translational equilibrium but not in rotational equilibrium.

For translational equilibrium

$$T\cos\alpha=\mu N$$

and $$T\sin\alpha+N=mg.$$

$\therefore$ $$T\cos\alpha=\mu(mg-T\sin\alpha)$$

or $$T=\frac{\mu mg}{\cos\alpha+\mu\sin\alpha}.$$

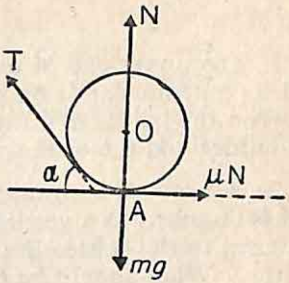


Fig. 3.27

Put $\mu=\tan\lambda$ where λ is the angle of friction.

$$T=\frac{mg\sin\lambda}{\cos(\alpha-\lambda)}.$$

Obviously T is minimum when $\cos(\alpha-\lambda)$ is maximum, that is,

$$\cos(\alpha-\lambda)=1,\quad \text{or}\quad \alpha=\lambda.$$

$$\therefore\quad T_{min}=mg\sin\lambda.$$

$$\because\quad \mu=\tan\lambda,\quad \therefore\quad \sin\lambda=\frac{\mu}{\sqrt{1+\mu^2}}.$$

$$\therefore\quad T_{min}=\frac{\mu mg}{\sqrt{1+\mu^2}}.\quad \textit{Ans.}$$

$$\pi=\lambda=\tan^{-1}\mu.\quad \textit{Ans.}$$

10. A jack has a screw with a pitch of 4 mm and a handle half metre long. The effect of friction is such that an extra effort of 0·004 part of the load is required on the handle. Calculate the least effort required on the end of the handle (a) to raise and (b) lower a load of 5000 N.

Sol. Let E be the effort required. Then the effective effort is $(E-\cdot004\times5000)=(E-20)$, because frictional force effectively reduces the applied effort.

Work done by effort $=(E-20)\times(2\pi\times\frac{1}{2})=\pi(E-20)$.

Work done by load $=5000\times(-\frac{4}{1000})=-20$.

Here work done by load is negative because its displacement is against its direction.

$\therefore$ Net work done by all external forces $=\pi(E-20)-20$.

According to 'work-energy theorem' this must be zero because the least effort is needed to just raise the load and not to generate velocity in it.

$$\therefore\quad \pi(E-20)-20=0,\quad \text{or}\quad E=20+\frac{20}{\pi}=26{\cdot}4\text{ N}.\quad \textit{Ans.}$$

When the load is lowered, work done by it $=+20$.

$$\therefore\quad \pi(E-20)+20=0$$

or $$E=20-\frac{20}{\pi}=20-6{\cdot}4=13{\cdot}6\text{ N}.\quad \textit{Ans.}$$

EXERCISES

1. The upper end of a ladder rests against a smooth vertical wall and its bottom stands on a rough floor. The coefficient of friction between the ladder and the floor is μ. Calculate the angle α between the ladder and the wall at which the former will be in equilibrium.

2. A heavy homogeneous sphere is suspended from a string whose end is attached to a vertical wall. The point at which the string is fastened to the sphere lies on the vertical line passing through its centre. What should be the cofficient of friction between the sphere and the wall for the sphere to remain in equilibrium ?

3. A homogeneous chain of length l lies on a table. What is the maximum length l_1 of the chain that can hang over the table if the coefficient of friction is μ ?

4. A boat on a lake is perpendicular to the shore and faces it with its bow. The distance between the bow and the shore is 1 m. At the initial moment the man was stationary at the bow side. He steps from the bow side to the stern. Will the boat reach the shore if it is 2 m long ? The mass of the boat $M = 140$ kg and that of the man $m = 40$ kg.

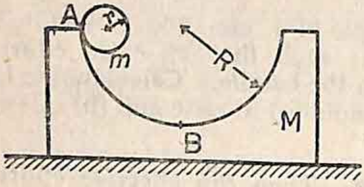


Fig. 3.28

5. A block of mass M with a semi-circular track of radius R, rests on a horizontal frictionless surface. A uniform cylinder of radius r and mass m is released from rest at the top point A (see figure). The cylinder slips on the semi-circular frictionless track. How far has the block moved when the cylinder reaches the bottom (point B) of the track ? How fast is the block moving when the cylinder reaches the bottom of the track ? **(IIT 1983)**

6. A stationary pulley carries a rope whose one end supports a ladder with a man and the other end the counterweight of mass M. The man of mass m climbs up a distance l' with respect to the ladder and then stops. Neglecting the mass of the rope and the friction in the pulley axle, find displacement l of the centre of mass of this system.

7. A horizontally directed force equal to the weight of a heavy roller of radius R is applied to its axis to lift it onto a rectangular step. Find the maximum height of the step.

8. A sphere of weight 3 kgf lies on two inclined planes forming angles 30° and 60° with the horizontal. Determine the pressure exerted on each plane if there is no friction between the sphere and one of the planes.

9. A raft of mass M with a man of mass m aboard stays motionless on the surface of a lake. The man moves a distance l' relative to the left with velocity $v'(t)$ and then stops. Assuming the water resistance to be negligible, find :

(a) the displacement of the raft l relative to the shore;

(b) the horizontal component of the force with which the man acted on the raft during the motion.

10. From a circular disc of radius r a square is cut out with a radius as its diagonal; show that the cg of the remainder is at a distance $\frac{r}{4\pi-2}$ from the centre.

11. A ladder is in equilibrium with one end resting on the ground and the other end against a vertical wall. If the coefficient of friction between the ground and the end of ladder be μ and that between the wall and the other end is μ', show that the inclination of the ladder to the horizontal is given by

$$\tan\theta=\frac{1-\mu\mu'}{2\mu}.$$

12. How high can a particle rest inside a hollow sphere of radius a and the coefficient of friction be $\mu=\frac{1}{\sqrt{3}}$?

13. A block of mass $M=2{\cdot}8$ kg is resting on a horizontal smooth surface as shown. The sides AB and BC are perfectly smooth each of length $l=39{\cdot}2$ em and each inclined at $\alpha=30^\circ$ to the horizontal. A particle of mass $m=0{\cdot}95$ kg starts sliding down along AB from rest from A. The planes AB and BC join at B in a smooth curved surface not exerting any resistance to motion.

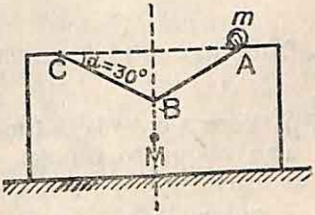


Fig. 3.29

Show that block M executes horizontal oscillatory motion. Find the general expression for its period of oscillation and then calculate its numerical value.

TEST YOURSELF

1. Answer the following questions :

(a) Define centre of gravity and centre of mass. Carefully distinguish between them. 5

(b) Show that effect of applying a couple on a body is always to turn the body. 5

(c) Find the dimensions of moment. 2

(d) Give an example to show that the following statement is false : Any two forces acting on a body can be combined into a single force that would have the same effect. 3

(e) Does a machine save our time ? Explain it with axle and wheel as example. 5

2. Answer the following objective questions. 3×7=21

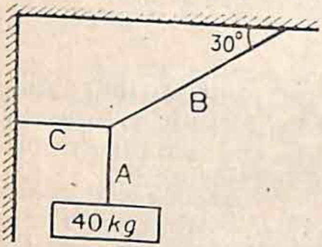


Fig. 3.30

(i) A weight of 40 kg is suspended as shown in the figure. The tension in the cord *B* is
(a) 80 kgf, (b) 20 kgf,
(c) 40 kgf, (d) 60 kgf.

(ii) A cube is in limiting equilibrium on an inclined plane forming angle 30° to the horizontal. The line of action of the normal reaction of the plane on the cube is (a) at the centre of the cube, (b) at a distance $\frac{\sqrt{3}a}{2}$ from the centre down the plane, (c) at a distance $\frac{\sqrt{3}a}{2}$ up the plane from the centre, (d) at a distance $\frac{a}{2\sqrt{3}}$, from the centre down the plane.

(iii) When a cube is in limiting equilibrium on an inclined plane, it is also about to topple. The coefficient of friction between the cube and the plane is (a) 1, (b) 1/2, (c) $1/\sqrt{2}$, (d) $1/\sqrt{3}$.

(iv) Two masses m_1 and m_2 are separated by a distance l. The distance of their centre of mass from m_1 is

(a) $\frac{m_1 l}{m_1+m}$, (b) $\frac{m_2 l}{m_1+m_2}$, (c) $\frac{m_1}{m_2}l$, (d) $\frac{m_2}{m_1}l$.

(v) When a body is in stable equilibrium, (a) its potential energy is maximum, (b) potential energy is zero, (c) potential energy is minimum, (d) potential energy fluctuates with time.

(vi) The potential energy of a system is given by $U=-U_0\cos\theta$ where θ is the angle made by the system with a fixed reference line. The body is in stable equilibrium when (a) $\theta=0°$, (b) $\theta=\pi/2$, (c) $\theta=\pi/4$, (d) $\theta=\pi$.

(vii) If A_i is the ideal mechanical advantage of a machine, A_r is its real mechanical advantage, then its efficiency is (a) A_iA_r, (b) A_i+A_r, (c) A_i/A_r, (d) A_r/A_i.

3. Answer the following : $3\times3=9$

(i) When a tradesman using a balance of unequal arms weighs half of the total amount on alternate pans, he cheats himself, not the customer. Is it true or false ?

(ii) A boat is likely to capsize if persons in the boat stand up. Why ?

(iii) Which one easier to turn—a log or a bench of equal weight and length. The two have the same coefficient of friction with the ground ?

Answer any five of the following. $5\times10=50$

4. A body of mass m is placed on the incline of a wedge of mass M at the highest point of it. The body is released. If there is no friction and l is the length of the incline, determine the distance by which the wedge moves along horizontal plane when the body reaches the bottom of the incline.

5. (a) A circular section of radius r is cut out of a uniform disc of radius R, the centre of the hole being $R/2$ from the centre of the original disc. Locate the centre of gravity of the resulting body.

(b) A beam of weight W is cerried by three men, one man at one end and the other two supporting the beam between them on a cross piece so that the load is equally divided among the three men. Find where the cross-piece is placed. Neglect the mass of the cross-piece.

6. (a) A flexible chain of weight W hangs between two fixed points, A and B, at the same level (see figure). Find the force exerted by the chain on each end point and the tension in the chain at the lowest point.

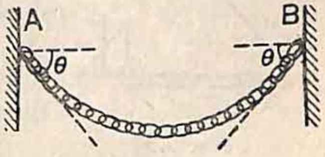


Fig. 3.31

(b) A thin bar AB, whose weight can be neglected, is suspended by strings from the two points P and Q (see fig.). The bar carries masses of 10 kg and 5 kg as shown. Find the tensions in the strings and the angle θ, if the system is in equilibrium.

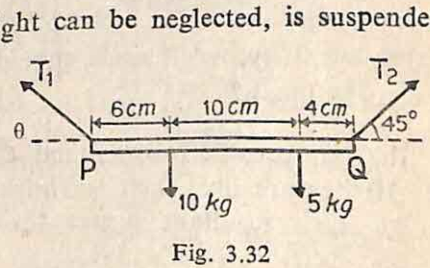


Fig. 3.32

7. (a) A cubical box rests on a rough horizontal rectangular plank whose coefficient of friction is 0·4. The edge AB of the plank is parallel to an edge of the box. The plank is slowly raised about

its edge AB and its inclination is slowly increased. Find whether the box will slide first or topple.

(b) A gate 3 m long and 2 m high weighs 60 kg. Its centre of gravity is at its centre, and it is hinged at A and B. To relieve the strain on the top hinge a wire CD is connected as shown in the figure. The tension in CD is increased until the horizontal force at hinge A is zero. What is the tension in the wire CD? What is the combined vertical force exerted by hinges A and B?

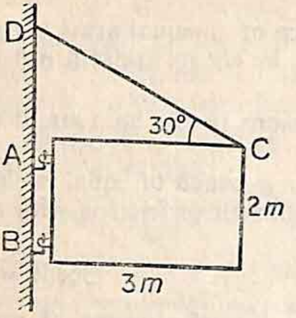


Fig. 3.33

8. (a) A man weighing 80 kg sits on a platform suspended from a movable pulley raises himself by a rope passing over a fixed pulley. Assuming no friction losses, find the force he must exert.

(b) The operator of a jackscrew exerts a 40 kgf force through 2·4 m to raise a 600 kg load by 5 cm. Find: output, input, efficiency, actual MA; ideal MA.

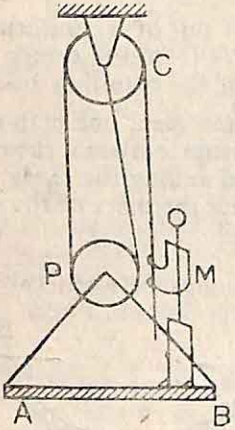


Fig. 3.34

9. (a) On a pulley system, 60% efficient, the effort of 40 kgf moves 60 m to raise the weight by 9 m. Find the ideal MA. At 100% efficiency, what loads should be lifted? At 60%, what load is actually lifted?

(b) Two parallel forces P and Q act respectively at points A and B. If they are like, their resultant passes through C and if they are unlike, their resultant passes through D. If $P > Q$, show that

$$CD = \frac{2PQ}{P^2 - Q^2} \times AB.$$

Full marks 100

Your score....

Answers to Exercises

1. $\tan^{-1}\mu$. **2.** $\mu=1$. **3.** $l_1=\dfrac{\mu l}{\mu+1}$. **4.** $\cdot 56$ m.

5. $\dfrac{m(R-r)}{M+m}$. 6. $\dfrac{ml'}{2M}$. 7. $\cdot 3R$. **8.** 3 kgf; $\sqrt{3}$ kgf.

9. $l=\dfrac{ml'}{M+m}$; $F=\dfrac{mM}{m+M}\cdot\dfrac{dv'}{dt}$. **12.** $\cdot 134a$.

13. $4\sqrt{\dfrac{2l(M+m\sin^2\alpha)}{(M+m)g\sin\alpha}}$; 1·44 s.

Answers to Test Questions

2. (i) (a), (ii) (d), (iii) (a), (iv) (c), (v) (a), (vi) (a), (vii) (d)

3. See solutions. 4. $\dfrac{ml\cos\theta}{m+M}$. 5. $\dfrac{r^2R}{2(R^2-r^2)}$; $\dfrac{l}{4}$.

6. (a) $F=\dfrac{W}{2\sin\theta}$; $T=\dfrac{W}{2}\cot\theta$;

(b) $T_1=10{\cdot}6$ kgf; $T_2=9{\cdot}9$ kgf; $\theta=48°48'$.

7. (a) The box will slide first and then topple. (b) 46·1 k

8. (a) 27 kgf. (b) 940·8 J; 294·0 J; ·31; 15; 48.

9. (a) 4; 6·7; 268 kg, 160 kgf.

Solutions to Exercises

1. Let $2l$ be the length of the ladder.

Considering vertical and horizontal linear equilibrium

$$W=F$$
$$\mu N=N'.$$
$$\therefore\quad \mu W=N'.$$

Considering rotational equilibrium about the end B

$$Wl\sin\alpha=N'l\cos\alpha$$

or $\qquad \tan\alpha=\dfrac{N'}{W}=\dfrac{\mu W}{W}$

or $\qquad \tan\alpha=\mu,$

or $\qquad \alpha=\tan^{-1}\mu.$ *Ans.*

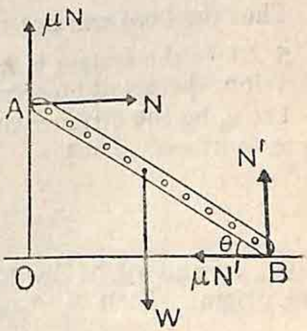


Fig. 3.35

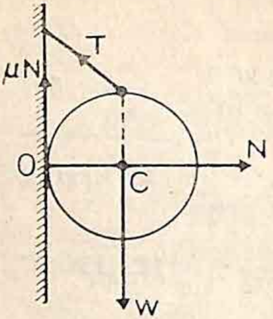


Fig. 3.36

2. Considering rotational equilibrium about the point of suspension

$\mu N \times a = Na$ where a = radius of the sphere

or $\mu = 1$. *Ans.*

3. When the limiting frictional force on the part resting on the table is equal to the weight of the hanging part, the maximum length will hang over.

$\therefore\ l_1 mg = \mu(l - l_1)mg$ where m = mass per unit length

or
$$l_1 = \frac{\mu l}{\mu + 1}. \quad Ans.$$

4. Since the system (man+boat) is free from external force in the horizontal direction, the position of the centre of mass will not change.

$$x_{cm} = \frac{40 \times 1 + 140 \times 2}{40 + 140} = \frac{320}{180} = 1{\cdot}78 \text{ m.}$$

Let x be the distance of the bow from the shore when the man walks to the stern. Then

$$x_{cm} = \frac{40 \times (x+2) + 140(x+1)}{180}.$$

$$\therefore\ 1{\cdot}78 = \frac{180x + 220}{180}, \text{ or } x = {\cdot}56 \text{ m.}$$

Thus the boat will not reach the shore. *Ans.*

5. Since the system is free from external force in the horizontal direction, there will no change in the position of its centre of mass.

Let x_1 be the distance of the centre of M from any fixed point on the surface. Then

$$x_{cm} = \frac{Mx_1 + m(x_1 - R + r)}{M + m}.$$

Let x_1' and x_2' be the positions of the c.g.'s M and m from the same origin. Then $x_1' = x_2'$.

$$\therefore\ x_{cm} = \frac{Mx_1' + mx_1'}{M + m} = x_1'.$$

$$\therefore \quad \frac{Mx_1+m(x_1-R+r)}{M+m}=x_1'$$

or $$(M+m)x_1-m(R-r)=(M+m)x_1'$$

or $$(M+m)(x_1-x_1')=m(R-r)$$

$$\therefore \quad x_1-x_1'=\frac{m(R-r)}{M+m}. \quad Ans.$$

6. Let x_1 and x_2 be the distances of the counterweight and ladder (man is assumed to be at the centre of the ladder) from the level of the pulley. Then

$$x_{cm}=\frac{Mx_1+Mx_2}{M+M}=\frac{x_1+x_2}{2}.$$

Let the centre of the ladder go down by z.

Then distance of its centre of mass $=x_2+z$.
Distance of the centre of the counterweight $=(x_1-z)$.

Let x_3 be the distance of the man from the pulley.

Then $l'=(x_2+z)-x_3$ ($\because$ relative displacement = vector difference of absolute position vectors)

or $x_3=(x_2+z)-l'$.

$$\therefore \quad x'_{cm}=\frac{M(x_1-z)+(M-m)(x_2+z)+mx_3}{M+M-m+m}$$

$$=\frac{M(x_1-z)+(M-m)(x_2+z)+m(x_2+z-l')}{2M}$$

$$=\frac{M(x_1-z)+M(x_2+z)-ml'}{2M}$$

$$=\frac{M(x_1+x_2)-ml'}{2M}=\frac{x_1+x_2}{2}-\frac{ml'}{2M}.$$

$\therefore$ l (displacement of the centre of mass of the system)

$$=x_{cm}-x'_{cm}=\frac{x_1+x_2}{2}-\frac{x_1+x_2}{2}+\frac{ml'}{2M}$$

or $$l=\frac{ml'}{2M}. \quad Ans.$$

7. Let h be the maximum height of the step. Then distance of the edge O of the step from the vertical through centre of the roller is

$$\sqrt{R^2-(R-h)^2}=\sqrt{2Rh-h^2}.$$

Considering O as 'moment centre'.

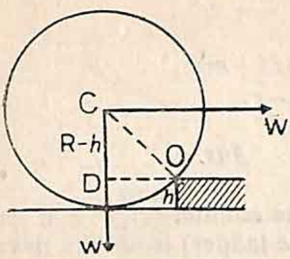


Fig. 3.37

we have for riding of the roller over the step,

$$W(R-h)=W\sqrt{2Rh-h^2}$$

or $(R-h)^2=2Rh-h^2$

or $R^2-2Rh+h^2=2Rh-h^2$

or $2h^2-4Rh+R^2=0$

or $(h-R)^2=\frac{R^2}{2}$

or $h-R=\pm\frac{R}{\sqrt{2}}$

or $$h=R\pm\frac{R}{\sqrt{2}}.$$

$\because\ h<R,\quad \therefore\ h=R-\frac{R}{\sqrt{2}}=\cdot 3R.$ *Ans.*

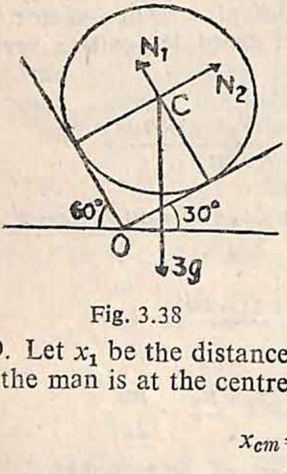


Fig. 3.38

8. We have

$N_1\cos 30^\circ+N_2\cos 60^\circ=3g$

and $N_1\sin 60^\circ=N_2\sin 30^\circ$

or $N_1\frac{\sqrt{3}}{2}+N_2\frac{1}{2}=3g$

or $N_1\sqrt{3}+N_2=6g$

and $N_1\sqrt{3}=N_2.$

$$\left.\begin{array}{l}\therefore\ N_2=3g=3\text{ kgf}\\ \text{and } N_1=\sqrt{3}g=\sqrt{3}\text{ kgf}\end{array}\right\}.\ \textit{Ans.}$$

9. Let x_1 be the distance of the centre of the raft from the shore and the man is at the centre. Then

$$x_{cm}=\frac{Mx_1+mx_1}{M+m}=x_1.$$

Let x_1' and x_2' be the distances of the centre of the raft and the man from the shore when the latter moves l' relative to the first. Then

$$l'=x_1'-x_2',\quad \text{or}\quad x_2'=x_1'-l'.$$

Now $$x'_{cm}=\frac{Mx_1'+mx_2'}{M+m}.$$

Since the system is free from external force along the horizontal, entre of mass remains stationary.

$$\therefore\quad x_{cm}=x'_{cm}.$$

or $$x_1=\frac{Mx_1'+mx_2'}{M+m}$$

or $$(M+m)x_1=Mx_1'+m(x_1'-l')$$

or $$ml'=(M+m)(x_1'-x_1)$$

or $$x_1'-x_1=\frac{ml'}{M+m}, \quad \text{or} \quad l=\frac{ml'}{M+m}. \quad \textit{Ans.}$$

Differentiating, $$\frac{dl}{dt}=\frac{m}{M+m}\cdot\frac{dl'}{dt}$$

or v (absolute velocity)$=\frac{m}{M+m}v'$ where v' = relative velocity of man

or $$\frac{dv}{dt}=\frac{m}{M+m}\cdot\frac{dv'}{dt}.$$

Force exerted by the man on the raft

$$=M\frac{dv}{dt}=\frac{mM}{m+M}\cdot\frac{dv'}{dt}. \quad \textit{Ans.}$$

10. Let us take the centre of the circle as origin of the frame of reference and the diagonal of the square as x-axis.

Mass of the circle $=\pi r^2 m$ where m is the mass per unit area,

Mass of the square $=\left(\frac{r^2}{2}\right)m$

$\left(\because \text{ Each side of the square}=\frac{r}{\sqrt{2}}\right)$.

Mass of the remainder $=\pi r^2 m-\frac{r^2}{2}m.$

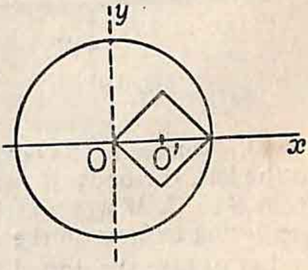


Fig. 3.39

Using the formula, $x_{cm}=\frac{m_1x_1+m_2x_2}{m_1+m_2}$

we have

$$0=\frac{\left(\frac{r^2}{2}m\right)\times\frac{r}{2}+r^2m(\pi-\frac{1}{2})\times x_2}{\pi r^2 m}, \quad (\because \quad x_{cm}=0)$$

or $$\tfrac{1}{4}r^3m+r^2m(\pi-\tfrac{1}{2})x_2=0$$

or $$r+(4\pi-2)x_2=0$$

or $$x_2=-\frac{r}{4\pi-2}. \quad \textit{Proved.}$$

The negative sign shows that it is on the other side of the centre.

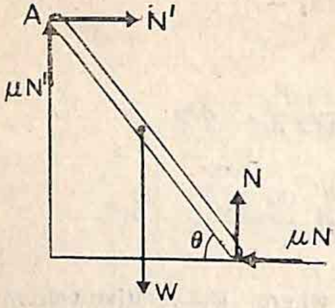


Fig. 3.40

11. Considering linear equilibrium along horizontal and vertical,

$$\mu N = N'$$

$$\mu' N' + N = W.$$

$$\therefore \quad \mu\mu' N = \mu' N' = W - N$$

$$\text{or} \quad (\mu\mu' + 1)N = W.$$

Considering rotational equilibrium about A (taking the length of the ladder $2l$),

$$W \times l\cos\theta + \mu N 2l\sin\theta = N2l\cos\theta$$

$$\text{or} \quad (1+\mu\mu')N\cos\theta + \mu N2\sin\theta = N2\cos\theta$$

$$\text{or} \quad (1+\mu\mu')\cos\theta + 2\mu\sin\theta = 2\cos\theta$$

$$\text{or} \quad 2\mu\sin\theta = (1-\mu\mu')\cos\theta$$

$$\text{or} \qquad \tan\theta = \frac{1-\mu\mu'}{2\mu}. \quad \textit{Proved.}$$

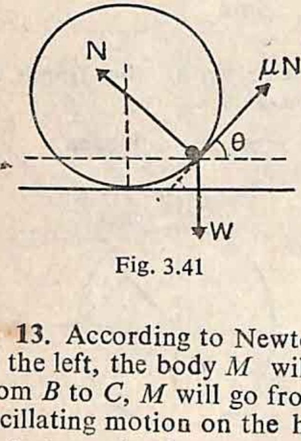


Fig. 3.41

12. Considering horizontal and vertical equilibrium we have,

$$N\sin\theta = \mu N\cos\theta$$

$$W = \mu N\sin\theta + N\cos\theta$$

$$\therefore \quad \tan\theta = \mu.$$

The required height

$$= h = a - a\cos\theta$$

$$= a\left(1 - \frac{1}{\sqrt{1+\mu^2}}\right)$$

$$= a\left(1 - \frac{\sqrt{3}}{2}\right) = \cdot 134a. \quad \textit{Ans.}$$

13. According to Newton's third law as the body m moves down to the left, the body M will go to the right. During its journey from B to C, M will go from right to left. This way M will perform oscillating motion on the horizontal smooth surface.

Let us take the dotted line through B as the reference. The initial distances of M and m from this line are 0 and $l\cos\alpha$.

∴ Distance of the centre of mass of the system $(M+m)$

$$= \frac{M.0 + ml\cos\alpha}{M+m}.$$

Let m come down by z along the plane when actual positions of M and m are x_M and x_m from the reference line. Then the actual displacement of M is x_M to the right and that of m is $(l\cos\alpha - x_m)$ to the left.

∴ Distance of the centre of mass from the dotted line

$$= \frac{Mx_M + mx_m}{M+m}.$$

As the system is free from external horizontal force, the position of the centre of the system must remain stationary.

$$\therefore \quad \frac{ml\cos\alpha}{M+m} = \frac{Mx_M + mx_m}{M+m}.$$

or $$ml\cos\alpha = Mx_M + mx_m \qquad \ldots \text{ (i).}$$

Since z is the displacement of m down the plane, $z\cos\alpha$ is the relative leftward horizontal displacement of m relative to M and $z\sin\alpha$ is the vertical downward displacement of m relative to M.

$\therefore \quad z\cos\alpha = (l\cos\alpha - x_m) - (-x_M)$, since relative displacement is the vector difference of the absolute displacements

and $\quad z\sin\alpha = y_m - 0 = y_m \quad (\because$ there is no vertical displacement of M).

Putting the value x_M in (i)

$$M(z\cos\alpha - l\cos\alpha + x_m) + mx_u = ml\cos\alpha$$

or $$Mz\cos\alpha + (M+m)x_m = (M+m)l\cos\alpha.$$

Differentiating w.r. to t, $\quad M\ddot{z}\cos\alpha = -(M+m)\ddot{x}_m$

or $$Ma\cos\alpha = -(M+m)a_m$$

where a is the relative acceleration of m relative to M and a_x is the absolute acceleration (horizontal) of m to the right or $(-a_x)$ is the leftward acceleration of m.

$$\therefore \quad a\cos\alpha = \left(1 + \frac{m}{M}\right) a_x \text{ (numerically)}$$

and $$a\sin\alpha = a_y.$$

The forces acting on m are : mg, downward and N reaction of the incline at an angle α with the upward vertical.

Considering horizontal and vertical motion of m

$$N\sin\alpha = ma_x = \frac{ma\cos\alpha}{\left(1 + \frac{m}{M}\right)} = \frac{mMa\cos\alpha}{M+m}$$

and $$mg - N\cos\alpha = ma_y = ma\sin\alpha.$$

Eliminating N between these two, we have

$$mg - \frac{mMa\cos^2\alpha}{(M+m)\sin\alpha} = ma\sin\alpha$$

or $$a = \frac{(M+m)g\sin\alpha}{M + m\sin^2\alpha}.$$

When m comes from A to B, its relative displacement is l.

$$\therefore \quad l = \tfrac{1}{2}at^2 \quad (\text{from } s = ut + \tfrac{1}{2}at^2)$$

or $$l = \tfrac{1}{2}\frac{(M+m)g\sin\alpha}{M + m\sin^2\alpha}.t^2$$

or $$t=\sqrt{\frac{2l(M+m\sin^2\alpha)}{(M+m)g\sin\alpha}}.$$

This is the time taken to make $\frac{1}{4}$th of an oscillation by M.

$$\therefore \quad T=4t=4\sqrt{\frac{2l(M+m\sin^2\alpha)}{(M+m)g\sin\alpha}}$$

$$=4\sqrt{\frac{2\times \cdot 392(2\cdot 8+\cdot 95\sin^2 30^\circ)}{(2\cdot 8+\cdot 95)\times 9\cdot 8\sin 30^\circ}}=1\cdot 44 \text{ s.} \quad \textit{Ans.}$$

Solutions to Test Questions

1. See Introductory Physics Part I for (a) and (b).

(c) ML^2T^{-2}.

(d) Two unlike equal parallel forces.

(e) No, a machine does not save our time. In an axle and wheel, the load is raised at a low speed.

2. (i) (a).

(ii) (d). In limiting equilibrium we have for 'linear equilibrium'

$$mg\sin 30^\circ=\mu N$$

and $$mg\cos 30^\circ=N$$

or $$\mu=\tan 30^\circ=\frac{1}{\sqrt{3}}.$$

Considering rotational equilibrium about the centre of mass,

$$\mu N\times\frac{a}{2}=N\times x$$

or $$x=\frac{a}{2\sqrt{3}}.$$

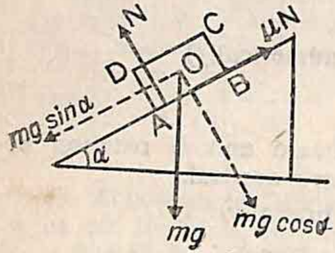


Fig. 3.42

(iii) (a). In the limiting equilibrium $\mu=\tan\alpha$. This relation is obtained from the consideration of limiting linear equilibrium. Since it is about to topple, the normal reaction is at the lower edge A.

$$\therefore \quad \mu N\times\frac{a}{2}=N\times\frac{a}{2}, \quad \text{or} \quad \mu=1.$$

(iv) (c). (v) (a).

(vi) (a). (vii) (d).

3. True. Let him weigh out $2W$ to a customer. Let W_1 and W_2 be the weights of material he gives away to the customer. If a and b are the lengths of the arms then

$$W_1a=Wb \quad \text{and} \quad Wa=W_1b.$$

He gives away (W_1+W_2) instead of $2W$.

$$\therefore \quad \text{Loss}=(W_1+W_2)-2W=W\left(\frac{a}{b}+\frac{b}{a}\right)-2W$$

$$=W\frac{(a-b)^2}{ab}.$$

This is always positive whether $a>b$ or $b>a$. Hence the tradesman always loses.

(ii) If persons stand up, the cg of the system (boat+passengers) is raised and so it is likely to capsize.

(iii) In case of a log the frictional force is distributed all along its length and so it is effective at the mid-way between the end where the force is applied and the centre of the log but in case of a bench the frictional force is at the end. So to turn the log torque of magnitude $\mu mg.\frac{l}{2}$ is needed and to turn the bench a moment of magnitude μmgl is needed. Hence it is easier to turn a log than a bench of the same weight.

4. Let x_1 be the distance of the lower edge A of the cube from a fixed point O. Then distance of m from O is $(x_1+l\cos\theta)$. Let b be the horizontal distance of the cm (centre of mass) of the wedge from A. Then the distance of the cm of the system from O is given by

$$x_{cm}=\frac{m(x_1+l\cos\theta)+M(x_1+b)}{m+M}.$$

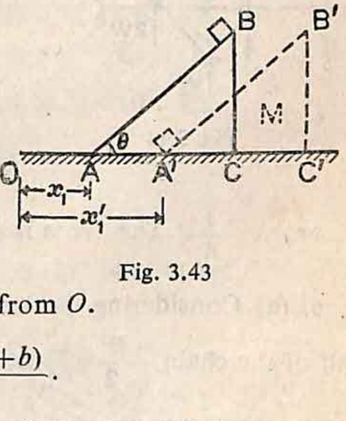


Fig. 3.43

Let x_1' be the distance of the edge A from O.

Then $$x'_{cm}=\frac{(mx_1'+M(x_1'+b)}{m+M}.$$

Since the system is free from external (horizontal) force, x_{cm} remains unchanged.

$$\therefore \quad x_{cm}=x'_{cm}$$

or $\quad mx_1+ml\cos\theta+Mx_1+Mb=mx_1'+Mx_1'+Mb$

or $\quad ml\cos\theta=m(x_1'-x_1)+M(x_1'-x_1)$

or $\quad x_1'-x_1=\dfrac{ml\cos\theta}{m+M}.$ *Ans.*

5. (a) Let us take the centre as origin of the frame of reference and radius of the hole as x-axis.

Mass of the disc $=\pi R^2m$ where $m=$ mass per unit area.

Mass of the section $=\pi r^2 m$ at a distance $\frac{R}{2}$.

Mass of the remainder $=\pi R^2 m-\pi r^2 m$.

Using the formula $x_{cm}=\frac{m_1 x_1+m_2 x_2}{m_1+m_2}$,

$$0=\frac{\pi r^2 m\times\frac{R}{2}+(\pi R^2 m-\pi r^3 m)x_2}{\pi R^2 m}$$

or $\frac{r^2 R}{2}+(R^2-r^2)x_2=0$, or $x_2=\frac{-Rr^2}{2(R^2-r^2)}$.

The negative sign shows that it is on the other side.

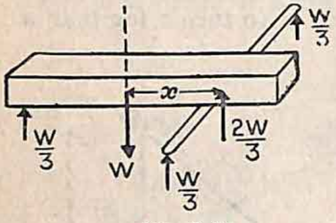


Fig. 3.44

$$\therefore\quad x_2=\frac{r^2 R}{2(R^2-r^2)}.$$

(b) Let x be the distance of the cross from the centre. Considering 'rotational equilibrium about the centre', we have

$$\frac{W}{3}\times\frac{l}{2}=\frac{2W}{3}\times\left(\frac{l}{2}-x\right)$$

where l is the length of the beam

or $x=\frac{l}{4}$. The cross is at a distance $\frac{l}{4}$ from the centre. *Ans.*

6. (a) Considering horizontal and vertical equilibrium of one half of the chain, $\frac{W}{2}=F\sin\theta$ and $T=F\cos\theta$.

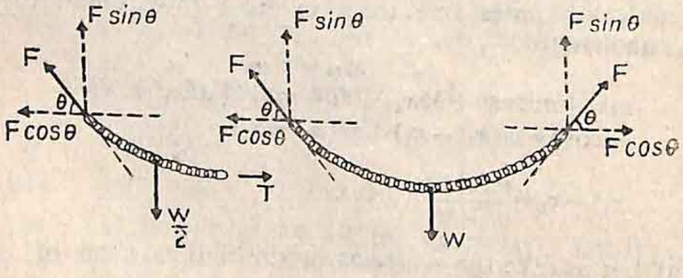


Fig. 3.45

Considering equilibrium of the entire chain

$$F\sin\theta+F\sin\theta=W.$$

$$\therefore\quad F=\frac{W}{2\sin\theta}.\quad \textit{Ans.}$$

$$\therefore\quad T=F\cos\theta=\frac{W}{2}\cot\theta.\quad \textit{Ans.}$$

(b) Considering horizontal and vertical linear equilibrium we have

$$T_1\cos\theta=T_2\cos45°,\quad \text{or}\quad T_1\cos\theta=\frac{T_2}{\sqrt{2}}$$

and $T_1\sin\theta+T_2\sin45°=15g$, or $T_1\sin\theta+\dfrac{T_2}{\sqrt{2}}=15g$;

$$\therefore\quad T_1\sin\theta+T_1\cos\theta=15g.$$

Considering 'rotational equilibrium about Q',

$$5g\times{\cdot}04+10g\times{\cdot}14=T_1\times{\cdot}20\sin\theta$$

$$15{\cdot}68=T_1\times{\cdot}20\sin\theta$$

or $$T_1\sin\theta=78{\cdot}4.$$

$$\therefore\quad T_1\cos\theta=15g-78{\cdot}4=147-78{\cdot}4=68{\cdot}6\text{ N}.$$

Squaring and adding,

$$T_1=\sqrt{78{\cdot}4^2+68{\cdot}6^2}=104\text{ N}=10{\cdot}6\text{ kgf}.$$

Dividing, $$\tan\theta=\frac{78{\cdot}4}{68{\cdot}6}=1{\cdot}14$$

$$\theta=48°48'.$$

$$\therefore\quad T_2=\sqrt{2}\,T_1\cos\theta=\sqrt{2}\times68{\cdot}6=97\text{ N}=9{\cdot}9\text{ kgf}.$$

7. (a) The box will slide when the inclination is

$$\tan^{-1}.4=21°48'.$$

It will topple when the reaction is at the lowest position A. Considering rotational equilibrium about the centre,

$$N\frac{a}{2}=\mu N\frac{a}{2}$$

or $$\mu=1$$

or $$\text{inclination}=\tan^{-1}\mu=\tan^{-1}1=45°.$$

So the box will slide first and then topple. *Ans.*

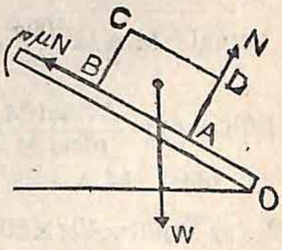


Fig. 3.46

(b) Considering 'translational equilibrium along the vertical and horizontal',

$$60g = V_2 + V_1 + T\sin 30^\circ \qquad \text{.. (i)}$$
$$H_2 = T\cos 30^\circ \qquad \text{.. (ii)}.$$

Considering 'rotational equilibrium about C',

$$(V_1+V_2)3 = 60g \times 1{\cdot}5 + H_2 \times 2$$

or $$3(V_1+V_2) = 90g + 2H_2 \qquad \text{.. (iii)}.$$

Combining (i) and (iii),

$$60g = \tfrac{1}{3}(90g + 2H_2) + T\sin 30^\circ$$

or $$180g = 90g + 2H_2 + 3T\sin 30^\circ$$

or $$90g = 2H_2 + 3T\sin 30^\circ \qquad \text{.. (iv)}.$$

Combining (ii) and (iv),

$$90g = 2T\cos 30^\circ + 3T\sin 30^\circ$$

$$T = \frac{90g}{2 \times {\cdot}866 + 3 \times {\cdot}5} = 27{\cdot}8g = 27{\cdot}8 \text{ kgf.}$$

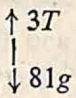
$$H_2 = T\cos 30^\circ = 27{\cdot}8 \times {\cdot}866 = 24{\cdot}1 \text{ kgf.}$$
$$V_1 + V_2 = 60g - T\sin 30 = 60g - 27{\cdot}8g \times {\cdot}5 = 46{\cdot}1 \text{ kgf.}$$

Thus, T (tension) $= 27{\cdot}8$ kgf,
H_2 (horizontal reaction at B) $= 24{\cdot}1$ kgf and
$V_1 + V_2$ (combined vertical force) $= 46{\cdot}1$ kgf. *Ans.*

8. (a) Considering free-body diagram of the system (plank + man),

$$3T = 80g$$

$$T = \frac{81g}{3} = 27g = 27 \text{ kgf.}$$



So the man must exert 27 kgf on the rope. *Ans.*

(b) Input work done by the effort $= 40g \times 2{\cdot}4 = 940{\cdot}8$ joule. *Ans.*
Output work done on the load $= 600g \times {\cdot}05 = 294{\cdot}0$ joule. *Ans.*

$\therefore$ Efficiency of the jack $= \dfrac{600g \times {\cdot}05}{40g \times 2{\cdot}4} = {\cdot}31$. *Ans.*

Actual M.A. $= \dfrac{600g}{40g} = 15$. *Ans.*

Efficiency $= \dfrac{\text{Actual M.A.}}{\text{Ideal M.A.}}$;

$\therefore$ Ideal M.A. $= 48$ *Ans.*

9. (a) Input $= 40g \times 60 = 2400g$ joule

${\cdot}6 = \dfrac{\text{output}}{2400\,g}$, or output $= 1440g$ joule $= W \times 9$

$\therefore$ W (load) $= 160g$

$\therefore$ Actual $\text{M.A.} = \frac{160g}{40g} = 4.$ *Ans.*

$$\cdot 6 = \frac{\text{Actual M.A.}}{\text{Ideal M.A.}};$$

$\therefore$ Ideal $\text{M.A.} = \frac{4}{\cdot 6} = 6{\cdot}7.$ *Ans.*

At 100% efficiency, ideal M.A. is effective.

$\therefore$ $6{\cdot}7 = \frac{\text{load}}{40}$, or Load = 268 kg. *Ans.*

At 60%, load = 160 kgf. *Ans.*

(b) The position of the resultant of two parallel forces is given by

$$x = \frac{F_1x_1 + F_2x_2 + \dots}{F_1 + F_2 + \dots}.$$

Let us take A as reference point.

$$AC = \frac{P \times 0 + Q \times AB}{P+Q}, \text{ or } AC = \frac{Q \times AB}{P+Q}$$

and $$AD = \frac{P \times 0 - Q \times AB}{P-Q}, \text{ or } AD = \frac{-Q \times AB}{P-Q}.$$

$$CD = AC - AD = \frac{Q \times AB}{P+Q} + \frac{Q \times AB}{P-Q} = \frac{2PQ}{P^2 - Q^2} \times AB. \text{ \textit{Proved.}}$$

LESSON 4

Topics : **Work, Power, Energy**
Principle of Conservation of Energy : Work-Energy Theorem
Collision of Bodies : Elastic and Inelastic

4.1 Work, Energy and Power

Work is done when a force (F) is displaced. The work done is

$$dW = F.dr.\cos\theta.$$

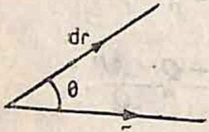


Fig. 4.1

Using vector notation,

$$dW = \vec{F} \,.\, \vec{dr}.$$

(i) When the force and displacement are in the same direction,

$$\theta = 0,\ \cos\theta = +1,$$

work done is positive.

(ii) When the force and displacement are in opposite directions, $\theta = 180°$, $\cos\theta = -1$, work done is negative.

(iii) When the displacement is perpendicular to the direction of the force, $\theta = 90°$, $\cos\theta = 0$. No work is done.

Work is a scalar quantity. Its unit is Joule. Power is the rate of doing work. Thus,

$$\text{Power} = \frac{\text{work done}}{\text{time taken}}.$$

The unit of power is Watt (=joule/sec).

Power of an agent is given by $P = \vec{F} \,.\, \vec{v}$ where F is the force applied by the agent and v is the velocity of body on which the agent applies the force.

The energy of a system is its capacity of doing work.

Mechanical energy may be of two types :

(i) Kinetic energy, (ii) Potential energy.

The kinetic energy of a system is

$$T = \tfrac{1}{2} m v^2.$$

The potential energy of a body in the gravitational field is given as

$$U=U_0+mgh$$

where U_0 = potential energy of the body at the ground level.

This is true only for object near the surface of the earth because g is uniform only near the surface.

The strain potential energy of a spring is given by $U=\frac{1}{2}kx^2$ where k is the 'force constant' of the spring and x is the change in length of the spring. This change in length may be either a compression or an extension.

4.2 Principle of Conservation of Energy

Conservative and Non-conservative Force. If the work done by a force in moving a body from one point to another depends only on the positions of the body and not on the process or path, the force is said to be conservative. The gravitational force, the spring force, elastic forces, the electric and magnetic forces are examples of conservative forces. If the work done depends on the path, it is said to be non-conservative force. Frictional force is an example of non-conservative force.

Work-Energy Theorem. Work done by external forces on a body is equal to the change of kinetic energy of the body. This is true both for constant forces and variable forces (variable in both magnitude and direction).

Principle of Conservation of Energy. Energy can neither be created nor destroyed by any process.

However, energy can be transformed from one form into another.

Note. The principle of conservation of energy and work-energy theorem are universal principles. They can be applied to all problems without any hesitation.

4.3 Collision of Bodies

Elastic Collision. When two bodies meet with certain relative speed they are said to collide with each other. In collision there is transfer of kinetic energy from one body to another. When the transfer of kinetic energy takes place in such a way that the total kinetic energy is conserved, the collision is said to be perfectly elastic, or simply elastic. When kinetic energy is not conserved, the collision is said to be inelastic. Further, in a collision, if one body is embedded into the other and the kinetic energy is not conserved, it is a completely inelastic collision. In inelastic and completely inelastic collisions there is always loss of some kinetic energy and this energy is converted into other forms, mostly heat.

The collision is said to be 'direct' or 'head-on' if the relative motion before and after collision is in the same direction; if not it is an 'oblique collision'.

In Solving Problems on Collision of Bodies Remember the following Points—

(i) Apply the principle of conservation of momentum. In one-dimensional direct collisions one equation is obtained by equating momentum before and after collision in the direction of motion. In two-dimensional collisions select line of impact as X-axis and line perpendicular to it as Y-axis and obtain two equations by equating momenta before and after collision along X and Y axes.

Remember that momentum is a vector quantity. It may be positive or negative depending on the direction. Choose any one direction as positive; the opposite will be negative.

(ii) If it is an elastic collision apply the principle of conservation of kinetic energy. For inelastic collisions apply the principle of conservation of energy to obtain an additional equation.

(iii) Remember there is no change in momentum along the common tangent to the colliding bodies.

Newton's Experimental Law. Coefficient of restitution—According to Newton the relative velocity of a body after collision is proportional to its relative velocity in the same direction before collision, with reversal of its sign.

Here, relative velocity means the relative velocity of any one of the colliding bodies (say A) with respect to the other colliding body (say B).

The constant of proportionality is called coefficient of restitution (e).

That is V_{AB} (after collision) $= -e \times V'_{AB}$ (before collision).

This is Newton's law of collision.

For elastic collisions, $e=1$. For inelastic collisions, $e<1$.

CRITICAL QUESTIONS

1. Two springs A and B are identical except that A is stiffer than B, that is, $k_A > k_B$. On which spring more work is done if they are stretched by the same force ?

Ans. We have $F = k_A x_A = k_B x_B$. $W_A = \frac{1}{2} k_A x_A^2$ and $W_B = \frac{1}{2} k_B x_B^2$.

$$\frac{W_A}{W_B} = \frac{k_A}{k_B} \times \frac{x_A^2}{x_B^2} = \frac{k_A}{k_B} \times \frac{k_B^2}{k_A^2} = \frac{k_B}{k_A}. \quad \therefore \quad W_B > W_A.$$

So more work is done on B.

2. Does the work done in raising a box onto a platform depend on how fast it is raised ?

Ans. No. Because work done depends only on the displacement and force.

3. Is their any gain in power in an ideal machine ?

Ans. No. Because in an ideal machine output and input are equal and the time in which input and output works are done is the same and so no gain in power.

4. Work done by external forces is always equal to the gain in kinetic energy. Is it always true ?

Ans. Yes. This is the universal work-energy theorem.

5. If there is a definite force and there is definite displacement of it, does that mean that work is definitely done by the force ? Explain.

Ans. No. The work done is given by three factors F, x and $\cos\theta$. That is, $W=Fx\cos\theta$. So if $F\neq 0$, $x\neq 0$, even then W can be zero if

$$\theta=\frac{\pi}{2},\ \frac{3\pi}{2}.$$

6. The work done by the resultant force is always equal to the change in kinetic energy. Is it possible that the work done by one of the component forces alone will be greater than the kinetic energy ? Explain.

Ans. Yes. If $\vec{F_1}$ and $\vec{F_2}$ are the component forces and $\vec{r_1}$ and $\vec{r_2}$ are their displacements then $\vec{F_2}\,.\,\vec{r_1}+\vec{F_2}\,.\,\vec{r_2}=T$ or $\vec{F_1}\,.\,\vec{r_1}-T=-\vec{F_2}\,.\,\vec{r_2}$. The dot product may be positive or negative. So if $\vec{F_1}\,.\,\vec{r_1}$ is +ve and $\vec{F_1}\,.\,\vec{r_1}$ is negative, $\vec{F_1}\,.\,\vec{r_1}>T$.

7. In a tug-of-war one team slowly gives way to the other. What work is being done and by whom ?

Ans. Work is done by the winning team on the other team. However, the net work done by all external forces (force exerted by the other team and the frictional force of the ground) is zero as the change in kinetic energy is zero.

8. Does the work done by a force depends on the frame of reference ?

Ans. Yes. To justify this let us consider a stationary frame S and a moving frame S' intially coincident. Let a body, initially at rest in the coincident frame, be observed by two observers in the two frames. Let further a constant force F start acting along x-axis on the particle at the moment S' starts separating out from S.

Then obviously $x=x'+ut$, where u is the speed of S'.

$$\therefore\quad \dot{x}=\dot{x}'+u$$

and $$\ddot{x}=\ddot{x}'$$

or $$a=a'.$$

So both observers observe the same acceleration. Let us now calculate the work done as observed from the two frames.

$$W=F.x=F(x'+ut)=Fx'+Fut=W'+Fut.$$

Obviously $$W \neq W'.$$

9 Work done by external forces on a body is equal to the change in kinetic energy of the body. This is called work-energy theorem. Does this theorem depend on the frame of reference ?

Ans. No. It is a universal principle. It holds in all the frames of reference.

10. When is a collision elastic ? Is total energy conserved in elastic collision ? Is kinetic energy ?

Ans. When kinetic energy of the system (the two colliding bodies) is conserved, collision is elastic. The principle of conservation of energy is a universal principle, it is obeyed in all phenomena. Yes, kinetic energy is conserved.

11. Is total energy conserved in inelastic collision ? Is kinetic energy ?

Ans. Yes, the universally true principle of conservation holds. No, kinetic energy is not conserved.

***12.** Consider a one-dimensional elastic collision between a given incoming body *A* and a body *B* initially at rest. How would you choose the mass of *B*, in comparison to the mass of *A* in order that *B* should recoil with (a) the greatest speed, (b) the greatest momentum, and (c) the greatest kinetic energy ?

Ans. By conservation of momentum,

$$m_1v_1=m_1v_1'+m_2v_2, \quad \text{or} \quad m_1(v_1-v_1')=m_2v_2.$$

By conservation of kinetic energy,

$$\tfrac{1}{2}m_1v_1^2=\tfrac{1}{2}m_1v_1'^2+\tfrac{1}{2}m_2v_2^2, \quad \text{or} \quad m_1(v_1-v_1')(v_1+v_1')=m_2v_2^2$$

$$\therefore \quad v_1+v_1'=v_2 \qquad (\because \quad v_1 \neq v_1')$$

$$v_1-v_1'=\frac{m_2}{m_1}v_2.$$

Adding $$2v_1=\left(1+\frac{m_2}{m_1}\right)v_2, \quad \text{or} \quad v_2=\frac{2v_1}{1+\frac{m_2}{m_1}}.$$

$\therefore$ (a) v_2 is maximum when m_2 is negligible in comparison to m_1.

(b) $$p_2=m_2v_2=\frac{2m_2v_1}{1+\frac{m_2}{m_1}}=\frac{2m_1m_2v_1}{m_1+m_2}=\frac{2m_1v_1}{1+\frac{m_1}{m_2}}.$$

Obviously p_2 is maximum when m_2 is large in comparison to m_1.

(c) $T_2=\frac{1}{2}m_2v_2^2=\frac{1}{2}m_2.\dfrac{4v_1^2}{\left(1+\dfrac{m_2}{m_1}\right)^2}$

$$=\frac{2m_2v_1^2m_1^2}{(m_1+m_2)^2}=\frac{2m_1^2v_1^2}{\left(\dfrac{m_1}{\sqrt{m_2}}+\sqrt{m_2}\right)^2}.$$

Now $$\left(\frac{m_1}{\sqrt{m_2}}+\sqrt{m_2}\right)^2=\left(\frac{m_1}{\sqrt{m_2}}-\sqrt{m_2}\right)^2+4m_1.$$

Obviously T_2 is maximum when $\left(\dfrac{m_1}{\sqrt{m_2}}+\sqrt{m_2}\right)^2$ is minimum but it is minimum when $\left(\dfrac{m_1}{\sqrt{m_2}}-\sqrt{m_2}\right)^2$ is minimum. This is a positive quantity. Its minimum value is zero.

$\therefore$ T_2 is maximum when $\dfrac{m_1}{\sqrt{m_2}}-\sqrt{m_2}=0$, or $m_1=m_2$ that is, when masses are equal.

13. If a particle of mass m_1 approaches another particle of mass m_2, initially at rest, with speed v_1, the speed of the centre of mass of the system is $v_{cm}=\dfrac{m_1v_1}{m_1+m_2}$ before collision takes place. Will the speed of centre of mass of the system change after collision ? Explain.

Ans. No, the speed of the centre of mass will not change because the speed of the centre of mass will change only when some external force acts on the system. During collision, only the two bodies interact and so there will be no change in the speed of the centre of mass of the system.

OBJECTIVE QUESTIONS

1. A heavy box of 40 kg is pushed through 20 m by two coolies on a railway-platform having 0·4 coefficient of friction with the box. The work done by the two coolies (take $g=10$ ms^{-2} and assume forces applied to be horizontal) is (a) +3200 J, (b) −3200 J, (c) +1600 J, (d) −1600 J.

Ans. (a).

2. In the above example, the work done by the frictional force is (a) +3200 J, (b) − 3200 J, (c) 1600 J, (d) −1600 J.

Ans. (b).

3. In the above example net work done by external forces on the box is (a) 3200 J, (b) 6400 J, (c) zero, (d) 1600 J.

Ans. (c).

4. A spherical ball of mass 4 kg, moving with 10 ms^{-1}, makes head-on elastic collision with another sphere of mass 12 kg at rest. The speed of the 12 kg body is (a) 4 ms^{-1}, (b) 5 ms^{-1}, (c) 6 ms^{-1}, (d) 3 ms^{-1}.

Ans. (b).

5. In the above example, the velocity of the 4 kg mass after collision is (a) $+5\ ms^{-1}$, (b) $-5\ ms^{-1}$, (c) 4 ms^{-1}, (d) 4 ms^{-1}.

Ans. (b).

6. A body of mass m_1 and speed v_1 makes head-on elastic collision with a body of mass m_2, initially at rest. The velocity of m_1 after collision is (a) $\frac{m_1+m_2}{m_1m_2}v_1$, (b) $\frac{m_1-m_2}{m_1+m_2}v_1$, (c) $\frac{2m_1v_1}{m_1+m_2}$, (d) $\frac{2m_2v_1}{m_1+m_2}$.

Ans. (b).

7. In the above example 6, the velocity of mass m_2 after collision is (a) $\frac{m_1+m_2}{m_1m_2}v_1$, (b) $\frac{m_1-m_2}{m_1+m_2}v_1$, (c) $\frac{2m_1v_1}{m_1+m_2}$, (d) $\frac{2m_2v_1}{m_1+m_2}$.

Ans. (c).

8. A body of mass m and speed v makes head-on collision completely inelastically with another body (of mass M) at rest. The velocity of M after collision is (a) $\frac{M-m}{M+m}v$, (b) $\frac{mv}{M+m}$, (c) $\frac{2mv}{M+m}$, (d) $\frac{2Mv}{M+m}$.

Ans. (b).

9. The potential energy of stretched spring of force constant k and extended through x is (a) kx^2, (b) $\frac{1}{2}kx^2$, (c) k^2x, (d) $\frac{1}{4}kx^2$.

Ans. (b).

10. The total gravitational potential energy of a body of mass m at a small height h is mgh. True or false.

Ans. False. The potential energy is U_0+mgh where U_0 is the potential energy in the reference level. 'mgh' is the potential energy relative to the reference level. The absolute value of the gravitational potential energy above the surface of the earth is

$$\left(-\frac{GMm}{R}\right)+mgh.$$

11. The potential energy of a particle is given by $U=\frac{a}{r^2}-\frac{b}{r}$ where a and b are positive constants, r is the distance from the centre of the field. The stable equilibrium position of the particle corresponds to the distance r_0 given by (a) $r_0=\frac{a}{2b}$, (b) $r_0=-\frac{a}{b}$, (c) $r_0=\frac{2a}{b}$, (d) $r_0=\frac{a}{b}$.

Ans. $U=\frac{a}{r^2}-\frac{b}{r}=\left(\frac{\sqrt{a}}{r}\right)^2-2.\frac{\sqrt{a}}{r}\cdot\frac{b}{2\sqrt{a}}+\left(\frac{b}{2\sqrt{a}}\right)^2-\left(\frac{b}{2\sqrt{a}}\right)^2$

(completing the square)

$$=\left(\frac{\sqrt{a}}{r}-\frac{b}{2\sqrt{a}}\right)^2-\frac{b^2}{4a}.$$

Obviously U is minimum when $\frac{\sqrt{a}}{r}-\frac{b}{2\sqrt{a}}=0$ or $r=\frac{2a}{b}$.

By calculus : $\frac{\partial U}{\partial r}=-\frac{2a}{r^3}+\frac{b}{r^2}$.

When U is minimum, $\frac{\partial U}{\partial r}=0$.

$$\therefore \quad \frac{-2a}{r^3}+\frac{b}{r^2}=0, \quad \text{or } r_0=\frac{2a}{b}.$$

So the answer is (c).

12. The work done by a force of 3 kgf in lifting a load of 1 kgf to a height of 5 m is (a) 98 joule, (b) 147 joule, (c) 9·8 joule, (d) 14·7 joule.

Ans. Work done by a force is given $W=\int \vec{F}\cdot d\vec{s}$ and it does not depend on the mass of the body on which it works. Here

$$W=3g\times 5\times\cos 0°=147 \text{ J}.$$

So the answers is (b).

13. A ball of mass m approaches a moving wall of infinite mass with speed v along the normal to the wall. The speed of the wall is u towards the ball. The speed of the ball after 'elastic' collision with wall is (a) $u+v$ away from the wall, (b) $2u+v$ away from the wall, (c) $u-v$ away from the wall, (d) $v-2u$ away from the wall.

Ans. Let the wall move to the right and let us refer all the velocities in this direction.

Before collision :

Relative velocity of the wall relative to the ball $= u - (-v) = u + v$.

After collision :

Relative velocity of the wall relative to the ball $= u - v'$ where v' is the velocity of the ball to the right after collision.

By Newton's law of collision,

$$u - v' = -(u + v), \quad \text{or} \quad v' = 2u + v.$$

So the answer is (b).

NUMERICAL QUESTIONS

1. A ball with a speed of 9 ms^{-1} strikes another identical ball such that after collision the direction of each ball makes an angle 30° with the original line of motion. Find the speeds of the two balls after the collision. Is the kinetic energy conserved in this collision process ?

(I.I.T. 1975)

Sol. Take line of approach as x-axis and a line perpendicular to it as y-axis.

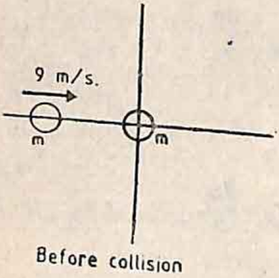


Fig. 4.2

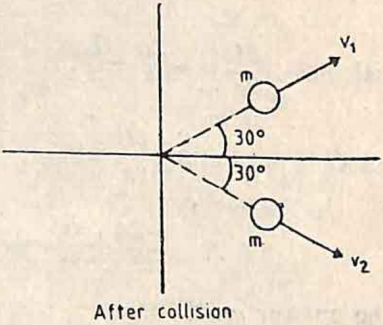


Fig. 4.3

From conservation of momentum along x- and y-axes,

$$m \times 9 = mv_1\cos30^\circ + mv_2\cos30^\circ$$

and $$0 = mv_1\sin30^\circ - mv_2\sin30^\circ$$

or $$v_1 + v_2 = 6\sqrt{3}$$

$$v_1 = v_2$$

$$\therefore \quad v_1 = v_2 = 3\sqrt{3}\ \text{ms}^{-1}. \quad \textit{Ans.}$$

Now $T_f - T_i = \frac{1}{2}m(3\sqrt{3})^2 \times 2 - \frac{1}{2}m.9^2 = -13{\cdot}5m$ joule.

So the kinetic energy is not conserved.

*2. A 0·5 kg body slides from the point A (see figure) on a horizontal track with an initial speed of 3 ms^{-1} towards a weightless horizontal spring of length 1 m and force constant 2 Nm^{-1}. The part AB of the track is frictionless and the part BC has the coefficients of static and dynamic friction 0·22 and 0·2 respectively. Find the total distance through which the block moves before it comes to rest completely ($g=10$ ms^{-2}), if $AB=2$ m and $BD=2{\cdot}14$ m. **(I.I.T. 1983)**

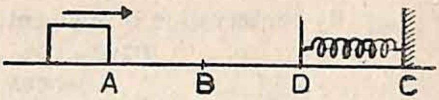


Fig. 4.4

Sol. The kinetic energy of the body $=\frac{1}{2}\times{\cdot}5\times3^2=2{\cdot}25$ J.

Work done by frictional force over $BD=\mu_k mg\times BD$

$$={\cdot}2\times{\cdot}5\times10\times2{\cdot}14$$
$$=2{\cdot}14 \text{ J.}$$

The rest of the energy $(2{\cdot}25-2{\cdot}14)=0{\cdot}11$ J is used in compressing the spring and doing work against friction.

Work done by elastic force $=-\frac{1}{2}kx^2=-\frac{1}{2}\times2\times x^2=-x^2$.

Work done by friction $=-\mu_k mgx=-{\cdot}2\times{\cdot}5\times10\times x=-x$.

Work done by gravity $=0$.

$\therefore$ Net work done by external forces $=-x^2-x+0=-(x^2+x)$.

By W-E theorem, $\quad -(x^2+x)=0-{\cdot}11$

or $\quad x^2+x={\cdot}11$

$$100x^2+100x-11=0$$
$$(10x+11)(10x-1)=0$$

$\therefore\quad x=\frac{1}{10}={\cdot}1$ m. The other value is not acceptable.

After this the block may describe some distance if it is pushed back by the spring. If the force of the spring exceeds the static limiting frictional force the block may describe some additional distance in the backward direction.

Limiting frictional force $=\mu_s mg={\cdot}22\times{\cdot}5\times10=1{\cdot}10$ N.

Force of the spring $=kx=2\times{\cdot}1={\cdot}2$ N.

So the spring cannot push back the block.

$\therefore$ Total distance described $=2+2{\cdot}14+{\cdot}1=4{\cdot}24$ m. *Ans.*

3. A 20 gm bullet pierces through a plate of mass $M_1=1$ kg and then comes to rest inside a second plate of mass $M_2=2{\cdot}98$ kg as shown. It is found that the two plates, initially at rest, now move with equal velocities. Find the percentage loss in the initial velocity of the bullet when it is between M_1 and M_2. Neglect any loss of material of the plates, due to the action of the bullet. **(I.I.T. 1979)**

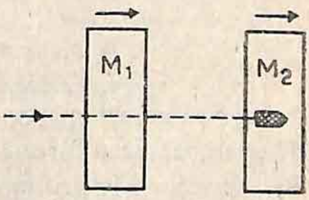


Fig. 4.5

Sol. By conservation of momentum principle,

$$\cdot 02v_0 = \cdot 02 \times v_1 + 1 \times v_2$$

where v_0 = initial velocity of the bullet
v_1 = velocity of the bullet between M_1 and M_2
v_2 = velocity of M_1

and $\cdot 02v_1 = (2 \cdot 98 + \cdot 02)v_2$.

Eliminating v_2, $v_1 = \frac{3}{4}v_0 = \cdot 75v_0$

$$\therefore \quad \% \text{ loss} \equiv \frac{v_0 - v_1}{v_0} \equiv \frac{v_0 - \cdot 75v_0}{v_0} = \cdot 25 \text{ or } 25\%. \quad \textit{Ans.}$$

4. A body of mass m is pushed with initial velocity v_0 up an inclined plane set at an angle α to the horizontal. The friction coefficient is equal to k. What distance will the body cover before it stops and what work do the friction forces perform over this distance ?

Sol. Work done by gravity $= -mgs\sin\alpha$ where s is the distance described along the plane.

Work done by frictional forces $= -kmg\cos\alpha . s$.

Net work done by external forces $= -mgs\sin\alpha - kmg\cos\alpha . s$.

By work-energy (W-E) theorem,

$$-mgs\sin\alpha - kmgs\cos\alpha = 0 - \tfrac{1}{2}mv_0^2$$

or

$$s = \frac{v_0^2}{2g(\sin\alpha + k\cos\alpha)}. \quad \textit{Ans.}$$

*5. A chain of mass $m = 0 \cdot 8$ kg and length $l = 1 \cdot 5$ m rests on a rough table so that one of its ends hangs over the edge. The chain slides off the table all by itself when $n = 1/3$ of its length hangs. What will be the total work done by the frictional forces acting on the chain by the time it slides completely off the table ?

Sol. The chain slips when 'nl' length hangs and $(l - nl)$ lies on the table.

$\therefore$ $\mu(l - nl) \times \rho g = nl\rho g$ where ρ is the mass per unit length of the chain.

$$\therefore \quad \mu = \frac{n}{1-n} = \frac{\frac{1}{3}}{1 - \frac{1}{3}} = \tfrac{1}{2} = \cdot 5.$$

Let x be the instantaneous length hanging down the table.

Then unbalanced force on the chain $= x\rho g - \mu(l - x)\rho g$.

By the second law of motion,

$$x\rho g = \mu(l - x)\rho g = l\rho \frac{dv}{dt}$$

or $$xg-\mu(l-x)g=l\frac{dv}{dx}\cdot\frac{dx}{dt}$$

or $$(1+\mu)xg-\mu lg=lv\frac{dv}{dx}\qquad\left(\because\quad v=\frac{dx}{dt}\right)$$

or $$\left(1+\frac{n}{1-n}\right)xg-\frac{n}{1-n}lg=lv\frac{dv}{dx}$$

or $$xg-nlg=(1-n)lv\frac{dv}{dx}$$

or $$gxdx-nlgdx=(1-n)lvdv.$$

Integrating, $\frac{1}{2}gx^2-nlgx=(1-n)l\frac{v^2}{2}+c.$

When $x=nl$, $v=0$.

$\therefore\ \frac{1}{2}gn^2l^2-n^2l^2g=c$ or $c=-\frac{1}{2}gn^2l^2$.

$$\therefore\ (1-n)l\frac{v^2}{2}=\frac{1}{2}gx^2-nlgx+\frac{1}{2}gn^2l^2$$

$$=\frac{1}{2}g(x^2+n^2l^2)-nlgx$$

or $(1-n)lv^2=g(x^2+n^2l^2)-2nlgx.$

When $x=l$, the velocity is given by

$$(1-n)lv^2=g(l^2+n^2l^2)-2ngl^2$$
$$=gl^2(1-n)^2$$

or $$v^2=gl(1-n).$$

$$\therefore\ \text{K.E.}=\tfrac{1}{2}mv^2=\tfrac{1}{2}mgl(1-n).$$

Instantaneous force due to gravity $=x\rho g$.

Elementary work done by gravity $=x\rho g dx$.

$\therefore$ Work done by gravity $=\displaystyle\int_{nl}^{l}x\rho g\,dx$

$$=\rho g\left(\frac{x^2}{2}\right)_{nl}^{l}=\tfrac{1}{2}\rho gl^2(1-n^2)=\tfrac{1}{2}mgl(1-n^2)\qquad(\because\quad m=l\rho).$$

Now by work-energy theorem,

$$W_{gravity}+W_{friction}=\text{change in kinetic energy.}$$

$$\therefore\ \tfrac{1}{2}mgl(1-n^2)+W_{friction}=\tfrac{1}{2}mgl(1-n)-0$$

or $$W_{friction}=\tfrac{1}{2}mgl(1-n)-\tfrac{1}{2}mgl(1-n^2)$$
$$=\tfrac{1}{2}mgl(1-n-1+n^2)=-\tfrac{1}{2}mgln(1-n).$$

$$\therefore\ W_{friction}=-\tfrac{1}{2}\times\cdot 8\times 9\cdot 8\times 1\cdot 5\times\cdot 5(1-\cdot 5)$$
$$=-\cdot 4\times 9\cdot 8\times 1\cdot 5\times\cdot 25=-1\cdot 3\text{ J.}$$ *Ans.*

*6. Two blocks A and B are connected to each other by a string and a spring; the string passes over a frictionless pulley as shown in the figure. Block B slides over the horizontal top surface of a stationary block C and the block A slides along the vertical side of C, both with the same uniform speed. The coefficient of friction between the surfaces of blocks is 0·2. Force constant of the spring is 1960 newton/m. If the mass of block A is 2 kg, calcutate the mass of block B and the energy stored in the spring. (I.I.T. 1982)

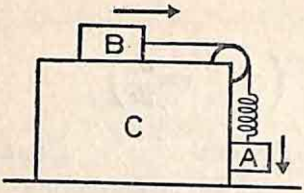


Fig. 4.6

Sol. Let m be the mass of B. From its 'free-body' diagram,

$$T-\mu N=m\times 0$$

where T= tension of the spring

and $N=mg$;

$$\therefore \quad T=\mu mg.$$

From the 'free-body' diagram of the spring,

$T-T'=0$ where T' is force exerted by A on the spring

or $T=T'=\mu mg$.

From the 'free-body' diagram of A,

$2g-(T'+\mu N')=2\times 0$ where N' is the normal reaction of the vertical wall of C on A

and $N'=2\times 0$ (as there is no horizontal acceleration of A)

$$\therefore \quad 2g=T'=\mu mg; \quad \text{or} \quad m=\frac{2g}{\mu g}=\frac{2}{\cdot 2}=10 \text{ kg.} \quad \textit{Ans.}$$

Tensile force on the spring $=T$ or $T'=\mu mg$

$$=\cdot 2\times 10\times 9\,8=19{\cdot}6 \text{ N.}$$

Now in a spring, tensile force = force constant × extension.

$$\therefore \quad 19{\cdot}6=1960x \quad \text{or} \quad x=\tfrac{1}{100} \text{ m.}$$

Now, U (energy of a spring) $=\frac{1}{2}kx^2$

$$=\tfrac{1}{2}\times 1960\times(\tfrac{1}{100})^2=\cdot 098 \text{ joule.} \quad \textit{Ans.}$$

7. A rope ladder with a length l carrying a man of mass m at its end is attached to the basket of a balloon of mass M. The entire system is in equilibrium in air. Find the work the man should do to climb into the basket. What is the velocity of the balloon if he climbs the ladder with a velocity v with respect to it ?

Sol. The external forces acting on the system when the man ascends are (i) F, buoyancy of air upwards, (ii) gravitational pull Mg, downwards, (iii) gravitational pull mg, downwards (iv) F_{man}, upwards.

When the man goes up by l, the balloon goes down by some distance, say, h.

For equilibruim of the system in the beginning (when the man is stationary)

$$F = Mg + mg.$$

When the man ascends :

$$\text{Work done by } F = -F.h.$$
$$\text{Work done by } Mg = Mg.h.$$
$$\text{Work done by } mg = -mg(l-h).$$

Net work done $= -F.h + Mgh - mg(l-h) + W_{man} = 0$ (since there is no change in kinetic energy of the system)

or $$W_{man} = mg(l-h) + Fh - Mgh$$
$$= mgl - mgh + Mgh + mgh - Mgh \qquad (\because \; F = Mg + mg)$$

or $$W_{man} = mgl.$$

This result can be obtained at once from a system fixed to the ladder.

$-mgl + W_{man} = 0$ or $W_{man} = mgl$. The work done by all other forces is zero because there displacements in this frame are zero.

Let v_m be the velocity (absolute) of the man upward and V is the absolute (i.e., relative to the ground) velocity of the balloon downward.

Relative velocity of man relative to balloon

$$= v_m - (-V) = v \text{ (given)}$$

or $$v_m = v - V.$$

By the conservation of momentum principle,

$$Mv_m + M(-v) = 0$$

or $$m(v-V) - MV = 0 \quad \text{or} \quad V = \frac{mv}{M+m}. \quad \textit{Ans.}$$

8. A system consists of two identical cubes, each of mass m, linked together by a compressed weightless spring of stiffness k. The cubes are also connected by a thread which is burnt at a certain moment. Find at what values of initial compression ϵ of the spring, the lower cube will bounce up after the thread is burnt through.

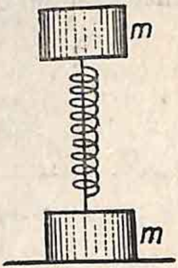


Fig. 4.7

Sol. Let us take the horizontal line through the c.g. of the lower cube as the reference level. Let l be the natural length of the spring.

Then initial energy of the system $=\frac{1}{2}k\epsilon^2+mg(l-\epsilon)$.

Let the spring extend by x after the thread is burnt.

The final energy of the system $=\frac{1}{2}kx^2+mg(l+x)$.

$$\therefore \quad \frac{1}{2}k\epsilon^2+mg(l-\epsilon)=\frac{1}{2}kx^2+mg(l+x)$$

(by principle of conservation of energy)

$$\text{or} \quad \frac{1}{2}kx^2+mgl+mgx-\frac{1}{2}k\epsilon^2-mgl+mg\epsilon=0$$

$$\text{or} \quad kx^2+2mgx-k\epsilon^2+2mg\epsilon=0$$

$$\text{or} \quad x=\frac{-2mg\pm\sqrt{4m^2g^2-4k(2mg\epsilon-k\epsilon^2)}}{2k}$$

$$\begin{aligned}\text{or} \quad kx&=-mg\pm\sqrt{(mg-k\epsilon)^2}\\ &=-mg\pm(mg-k\epsilon)\\ &=-k\epsilon \quad \text{or} \quad k\epsilon-2mg.\end{aligned}$$

$-k\epsilon$ is not acceptable.

$$\therefore \quad kx=k\epsilon-2mg.$$

The lower one will bounce up if $kx \geqslant mg$ (the weight of the lower cube)

$$\text{or} \quad k\epsilon-2mg\geqslant mg$$

$$\text{or} \quad k\epsilon\geqslant 3mg$$

$$\text{or} \quad \epsilon\geqslant\frac{3mg}{k}.$$

Thus for all values of ϵ greater than $\frac{3mg}{k}$ the lower cube will bounce up.

*9. Three identical discs A, B and C rest on a smooth horizontal plane. The disc A is set in motion with velocity v along the perpendicular bisector of the line BC joining the centres of the stationary discs. The distance BC between the centres of stationary discs B and C is n times the diameter of each disc. At what value of n will the disc A recoil, stop, move on after elastic collision?

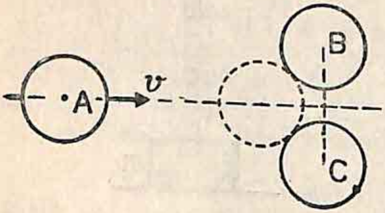


Fig. 4.8

Sol. Let the angle *BAC* be θ.

Then $\cos\theta = \dfrac{\frac{nd}{2}}{d} = \dfrac{n}{2}.$

Let v'' be the velocity of *B* and *C* after collision and v' be that of *A*.

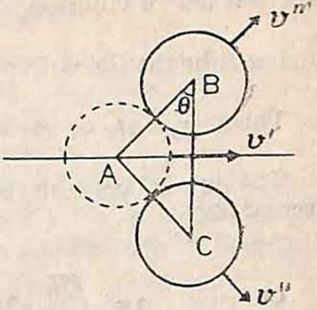


Fig. 4.9

Considering conservation of momentum along the 'common normal',

$$mv\sin\theta = mv'\sin\theta + mv''$$

or $(v-v')\sin\theta = v''$

or $(v-v')\dfrac{\sqrt{4-n^2}}{2} = v''.$

From conservation of kinetic energy as the collision is elastic we have

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}mv'^2 + 2.\tfrac{1}{2}mv''^2$$

or $$v^2 = v'^2 + 2v''^2$$

or $$v^2 - v'^2 = 2\times(v-v')^2\times\frac{4-n^2}{4}$$

or $$(v-v^2)(v+v') = 2(v-v')^2\times\frac{4-n^2}{4}$$

or $$v+v' = (v-v')\times\frac{4-n^2}{2}$$

or $$v' = -v\times\frac{2-n^2}{6-n^2}.$$

For recoil v' is to be negative.

∴ Condition is $2 > n^2$ or $n < \sqrt{2}$.

For stop, $v' = 0$, ∴ $2 - n^2 = 0$ or $n = \sqrt{2}$.

For on-move, $v' = +\text{ve}$; ∴ $2 < n^2$ or $n > \sqrt{2}$.

10. A plastic ball is dropped from a height of 1 m and rebounds several times from the floor. If 1·3 s elaspe from the moment it is dropped to the second impact with the floor, what is the coefficient of restitution ?

Sol. Let v_1 be the velocity acquired by the ball on reaching the floor. Then

$$v_1 = gt_1 \text{ and } v_1^2 = 2gh. \quad \therefore \quad t_1 = \frac{\sqrt{2gh}}{g} = \sqrt{\frac{2h}{g}}.$$

If v_2 is the velocity with which the ball bounces back, then by Newtons law of collision,

$$v_2 = ev_1.$$

Let t_2 be the time of upward journey after the first impact.

Them $v_2 = gt_2$ or $ev_1 = gt_2$ or $t_2 = \frac{e}{g}\sqrt{2gh} = e\sqrt{\frac{2h}{g}}$.

The interval between the moment of dropping and the moment of second impact is

$$\tau = t_1 + t_2 + t_2$$

or $$\tau = \sqrt{\frac{2h}{g}} + 2e\sqrt{\frac{2h}{g}} = (1+2e)\sqrt{\frac{2h}{g}}$$

∴ $$1{\cdot}3 = (1+2e)\sqrt{\frac{2\times 1}{9{\cdot}8}}.$$

Hence $e = 0{\cdot}98$. *Ans.*

***11.** Two blocks of equal mass m and connected by a spring of natural length l and spring constant k rest on an absolutely smooth horizontal surface. A third block of the same mass strikes with velocity v. Calculate the velocities of the blocks when the spring is compressed as much as possible.

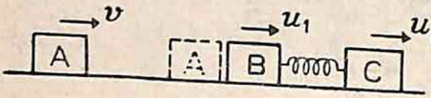


Fig. 4.10

Sol. Let A be the moving block and B and C are stationary blocks.

Since A and B are of equal mass, A is stopped 'dead' and B 'takes off' with its velocity. Now B and C move under their mutual action and reaction and so their momentum is conserved.

Let u_1 and u_2 be their instantaneous velocities when compression of the spring is x.

Then by conservation principle of momentum,

$$mv = m(u_1 + u_2) \quad \text{or} \quad u_1 + u_2 = v \text{ (a constant)}$$

and by the 'energy conservation principle',

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}mu_1^2 + \tfrac{1}{2}mu_2^2 + \tfrac{1}{2}kx^2$$

or $$v^2 = u_1^2 + u_2^2 + \frac{k}{m}x^2$$

$$= (u_1 + u_2)^2 - 2u_1u_2 + \frac{k}{m}x^2$$

or $$v^2 = v^2 - 2u_1u_2 + \frac{k}{m}x^2$$

or
$$u_1u_2=\frac{k}{2m}x^2.$$

Obviously compression (x) is maximum when u_1u_2 is maximum under the condition that their sum (u_1+u_2) is a constant.

Now $$(u_1+u_2)^2=(u_1-u_2)^2+4u_1u_2$$

or $$v^2=(u_1-u_2)^2+4u_1u_2$$

or $$4u_1u_2=v^2-(u_1-u_2)^2.$$

Obviously u_1u_2 is maximum when $(u_1-u_2)^2$ is minimum.

But it is a real positive quantity. Its minimum value is zero.

$$\therefore \quad (u_1u_2)_{max}=\frac{v^2}{4} \text{ when } u_1=u_2$$

and
$$x^2_{max}=\frac{2m}{k}(u_1u_2)_{max}=\frac{2m}{k}\cdot\frac{v^2}{4}$$

$$\therefore \quad x_{max}=\sqrt{\frac{m}{2k}}. \quad \textit{Ans.}$$

12. Two particles P and Q of mass 20 gm and 40 gm respectively are simultaneously projected from points A and B on the ground. The initial velocities of P and Q make 45° and 135° angles respectively with the horizontal AB as shown. Each particle has an initial speed of 49 m/s. The separation AB is 245 m. Both the particles travel in the same vertical plane and undergo a collision. After the collision, P retraces its path. Determine the position of Q when it hits the ground. How much time after the collision does the particle Q take to reach the ground ? $g=9{\cdot}8$ m/s^2. **(IIT 1983)**

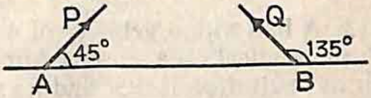


Fig. 4.11

Sol. Since the trajectory of a partcle does not depend on the mass of the particle, the two particles will have identical trajectory as their velocity of projection is the same and angle of projection with the horizontal is the same. Moreover, horizontal range of each

$$=\frac{u^2\sin 2\theta}{g}=\frac{49^2}{9{\cdot}8}=245=\text{given distance.}$$

Hence *they collide at the highest point of their trajectory. At the highest point the velocity of each is entirely horizontal*—*P's* velocity at that point is to the right and that of Q to the left. The value of the horizontal velocity of each at that point is

$$49\cos 45°=\frac{49}{\sqrt{2}}\ \text{ms}^{-1}.$$

Considering conservation of momentum at the highest point to the right,

$$20\times\frac{49}{\sqrt{2}}+40\left(-\frac{49}{\sqrt{2}}\right)=20\times\left(-\frac{49}{\sqrt{2}}\right)+40v$$

where v is the horizontal velocity of Q to the right after collision

or
$$-20\times\frac{49}{\sqrt{2}}=-20\times\frac{49}{\sqrt{2}}+40v$$

or
$$v=0.$$

So Q loses all of its momentum after collision and drops down to the ground from the highest point.

So Q hits the ground at a distance $\frac{245}{2}=122{\cdot}5$ m from its point of projection towards A.

Let t be the time of fall. This is also the time of ascent to the highest point.

$$\therefore\quad 49\sin 45^\circ=9{\cdot}8t$$

or
$$t=\frac{49}{9{\cdot}8\sqrt{2}}=3{\cdot}536 \text{ s.}\quad \textit{Ans.}$$

13. A ball with a velocity of 4 m/s impinges at an angle of 30° with the vertical on a smooth horizontal fixed plane. If the coefficient of restitution is 0·5, find its velocity and direction of motion after the first impact.

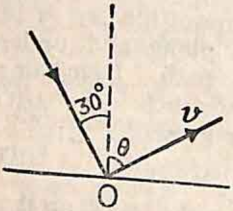


Fig. 4.12

Sol. Let the ball be reflected at angle θ with the vertical with speed v. Since the plane is smooth, there is '*no interaction*' along the horizontal and so momentum of the ball along this direction remains unchanged.

$$\therefore\quad 4\cos 60^\circ=v\cos(90^\circ-\theta)$$

or
$$2=v\sin\theta. \qquad \text{.. (i)}$$

Relative velocity of the ball relative to the plane before collision

$$=4\cos 30^\circ-0=4\cos 30^\circ.$$

Relative velocity of the ball relative to the plane after collision

$$=(-v\cos\theta)-0=-v\cos\theta.$$

By Newton's law of collision,

$$-v\cos\theta=-{\cdot}5\times 4\cos 30^\circ$$

or
$$v\cos\theta=2\times\frac{\sqrt{3}}{2}=\sqrt{3}. \qquad \text{.. (ii)}$$

Squaring and adding (i) and (ii),

$$v^2=4+3=7 \quad\text{or}\quad v=\sqrt{7} \text{ m/s.}$$

Dividing, $\tan\theta = \frac{2}{\sqrt{3}}$.

Thus the ball is reflected with velocity $\sqrt{7}$ m/s at $\tan^{-1}\frac{2}{\sqrt{3}}$ with the vertical. *Ans.*

EXERCISES

1. A block of mass 2 kg is forced against a horizontal spring of negligible mass, compressing the spring by an amount $x = 15$ cm. Upon releasing the block, it moves on a horizontal table top through a distance $s = 60$ cm before coming to rest. The force constant of the spring is 1960 N/m. What is the coefficient of friction between the block and the table ?

2. A 2-kg block is dropped from a height of 40 cm onto a spring whose force constant k is 1960 N/m. Find the maximum distance the spring will be compressed.

3. A block weighing 1 kg is released from rest at point A on a track which is one quadrant of a circle of radius 1·2 m. It slides down the track and reaches point B with a velocity of 3·6 m/s. From the point B it slides on a level surface a distance of 2·7 m to point C, where it comes to rest. (a) What was the coefficient of friction on the horizontal surface ? (b) How much work was done against friction as the body slid down the circular track from A to B ?

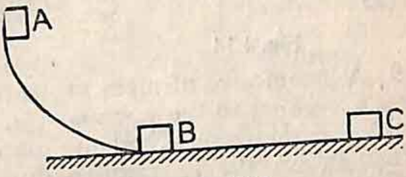


Fig. 4.13

4. Two particles A and B, having different but unknown masses, collide. A is initially at rest when B has a speed v. After collision B has a speed $v/2$ and moves at right angles to its original direction of motion. Find the direction in which A moves after collision. Can you determine the speed of A from the information given ? Explain.

5. Two vehicles A and B are travelling east and north, respectively, towards the same intersection where they collide and lock together. Before the collision A (900 kg in mass) is moving with a speed of 40 kph, and B (1200 kg in mass) has a speed of 60 kph. Find the magnitude and direction of the velocity of the interlocked vehicles immediately after collision.

*6. A particle falls from a height h upon a fixed horizontal plane. If e is the coefficient of restitution, show that the total distance described before the particle rebounds is

$$h\times\frac{1+e^2}{1-e^2}$$

and the total time taken is $\sqrt{\frac{2h}{g}}\times\frac{1+e}{1-e}$.

7. A steel ball of radius 2 cm is initially at rest on a horizontal frictionless surface. It is struck head-on by another steel ball of radius 4 cm, travelling with a velocity 81 cm/sec. Calculate the velocities of the two balls after collision. Assume the collision to be elastic. **(I.I.T. 1964)**

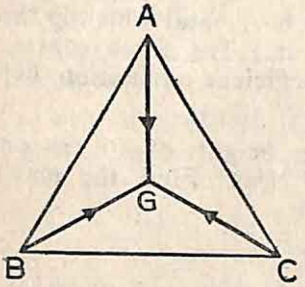


Fig. 4.14

*8. Three particles A, B and C of equal mass move with equal speed V along the medians of an equilateral triangle as shown in the figure. They collide at the centroid G of the triangle. After the collision, A comes to rest, B retraces its path with the speed V. What is the speed of C? **(I.I.T. 1982)**

9. A locomotive of mass m starts moving so that its velocity varies according to the law $v=a\sqrt{s}$, where a is a constant, and s is the distance covered. Find the total work done by all forces which are acting on the locomotive during the first t seconds after the beginning of the motion.

10. A paricle of mass m_1 experienced a perfectly elastic collision with a stationary particle of mass m_2. What fraction of the kinetic energy does the striking particle lose, if (a) it recoils at right angles to the original motion direction, (b) the collision is a head-on one?

11. A shell flying with velocity $v=500$ m/s bursts into three identical fragments so that kinetic energy of the system increases $n=1{\cdot}5$ times. What maximum velocity can one of the fragments obtain?

12. A particle of mass m_1 collides elastically with a stationary particle of mass m_2 $(m_1<m_2)$. Find the maximum angle through which the striking particle may deviate as a result of collision.

13. A particle 1 moving with velocity $v=10$ m/s experienced a head-on collision with a stationary particle 2 of the same mass. As a result of collision, the kinetic energy of the system decreased by $n=1\%$. Find the magnitude and direction of the velocity of the particle 1 after collision.

14. The bob of a pendulum of length 980 cm is released from rest with its string making an angle 60° with the vertical. It collides

with a ball of mass 20 gm resting on a smooth horizontal surface just at the position of rest of the pendulum. With what velocity will the bob move immediately after the collision ? What will be the maximum angular displacement of the bob after the collision ? The mass of the bob is 10 gm. **(I.I.T. 1966)**

*15. Two plates of masses m_1 and m_2 are connected by a spring. What force should be applied to the upper plate for it to raise the lower plate after the pressure is removed ? Consider the extreme case when the upper plate is of negligible mass.

16. A body of mass m moving with a velocity V in the X-direction collides with another body of mass M moving in the Y-direction with a velocity v. They coalesce into one body during the collision. Calculate—

(i) the direction and magnitude of the momentum of the final body.

(ii) the fraction of the initial kinetic energy transformed into heat during the collision in terms of the two masses. **(I.I.T. 1978)**

17. A ball moving on a horizontal frictionless plane hits an identical ball at rest with a velocity of 50 cm/sec. If the collision is elastic, calculate the speed imparted to the target ball, if the speed of the incident ball is 30 cm/sec after collision. Show that the two balls will move at right angles to each other after the collision. **(I.I.T. 1972)**

18. A cannon of mass M, located at the base of an inclined plane, shoots a shell of mass m in a horizontal direction with velocity v. To what vertical height does the cannon ascend the inclined plane as a result of recoil, if the angle of inclination of the plane is α and the coefficient of friction between the cannon and the plane is μ.

19. Two balls of masses m and $2m$ are suspended by two threads of the same length l and from the same point of a ceiling. The ball m is pulled aside through an angle α and released from rest, after imparting to it a tangential velocity v_0 towards the other stationary ball. To what height will the balls rise after collision if the collision is perfectly elastic ?

20. A 10 gm bullet is fired from a rifle horizontally into a 5 kg block of wood suspended by a string and the bullet gets embedded in the block. The impact causes the block to swing to a height of 5 cm above its initial level. Calculate the velocity of the bullet. **(I.I.T. 1967)**

21. A body of mass 2 kg makes an elastic collision with another body at rest and afterwards continues to move in the original direction but with one-fourth of its original speed. What is the mass of the struck body ?

22. A bullet of mass 10 gm strikes a ballistic pendulum of mass 2 kg. The centre of mass of the pendulum rises a vertical distance of 12 cm. Assuming the bullet remains in the pendulum, calculate its initial speed.

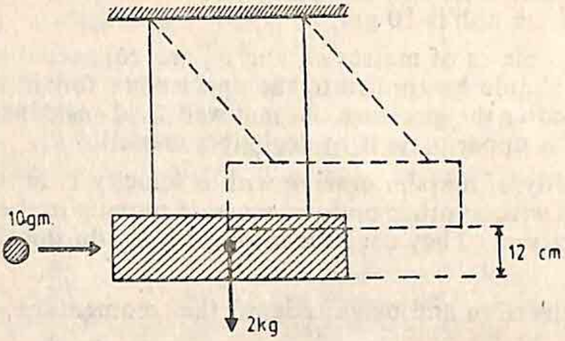


Fig. 4.15

23. A ball falls vertically from rest for 2 seconds and hits a smooth plane inclined at 30° to the horizontal. If the coefficient of restitution be $\frac{3}{4}$, find the time that elapses before it again hits the plane.

24. A massless horizontal spring of force constant 100 N/m is clamped to a vertical wall at one end and the other end is fixed to a block of 1 kg which rests on a frictionless horizontal table. A rifle bullet of mass ·1 kg travelling horizontally strikes the block and embeds itself in the block. The spring is compressed by ·1 m. Calculate the initial velocity of the bullet.

25. Two bodies of masses 2 kg and 1·5 kg, move towards each other and collide head-on inelastically. Before collision, the velocities of the bodies were $v_1 = 1$ m/s and $v_2 = 2$ m/s respectively. How long will the combined mass move after the collision if $\mu = 0{\cdot}05$?

26. A shell of mass $3m$ is moving horizontally through air with velocity u when an internal explosion causes it to separate into two parts of masses m and $2m$, which continue to move horizontally in the same vertical plane. If the explosion generates additional energy of amount $12mu^2$, prove that the two fragments separate with relative speed $6u$.

27. A particle A of mass m moving on a smooth horizontal surface collides with a stationary particle B of mass $2m$ directly, situated at a distance d from the wall. The coefficient of restitution between A and B and between B and the wall is $e = \frac{1}{4}$. Calculate the distance from the wall where they again collide. Assume the entire motion to take place along a straight line perpendicular to the wall.

***28** A wedge of mass M rests on an absolutely smooth horizontal surface. A block of mass m is placed on the wedge inclined at angle α to the horizontal. All the surfaces are frictionless. Assuming that the system was at rest at the initial moment, find the velocity of the wedge when the block slides down the plane through a vertical height h.

29. A steel ball of mass $m = 50$ gm falls from height $h = 1$ m on the horizontal surface of a mssive slab. Find the cumulative momentum imparted to the slab by the ball after numerous bounces, if the coefficient of restitution between the ball and the slab is $e = 0{\cdot}8$.

***30.** A wagon of mass M can move without friction along horizontal rails. A simple pendulum consisting of a sphere of mass m is suspended from the ceiling of the wagon by a string of length l. At the initial moment the wagon and the pendulum were at rest and the string was deflected through an angle α from the vertical. What will be the velocity of the wagon when pendulum string forms an angle β ($\beta < \alpha$) with the vertical and hence find the velocity of the wagon when the pendulum passes through its mean position.

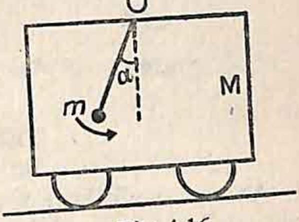


Fig. 4.16.

***31.** A homogenous chain of length $2l$ and mass M lies on an absolutely smooth table. A small part of the chain hangs from the table. At the initial moment of time the part of the chain lying on the table is held and released, after which the chain begins to slide off the table under the weight of the hanging end. Find the velocity of the chain when the length of the hanging part is equal to x ($x < l$). Also calculate the acceleration and the force with which it acts on the edge of the table.

***32.** A plate of mass m can freely move without friction between two smooth walls. The lower end of the plate rests on a smooth wedge lying on a horizontal surface (see figure of Ex. 39 in Lesson 2) lying on a smooth horizontal surface. The mass of the wedge is M. In the beginning the plate and the wedge are at rest. Find the velocity of the wedge at the moment plate descends by h.

33. Two balls of masses m and $2m$ are hanging on threads of length l fixed at the same point. The ball m is pulled to one side through an angle α and projected to $2m$ after imparting a tangential velocity v_0 to it. To what height will the balls rise after collision if

(a) the collision is perfectly elastic,

(b) completely inelastic.

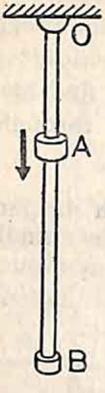


Fig. 4.17

34. A smooth rubber cord of length l whose coefficient of elasticity (spring constant) is k is suspended from the point O (see figure). The other end is fitted with a catch B. A small sleeve A of mass m starts falling from the point O. Neglecting masses of the cord and the catch, find the maximum elongation of the cord.

TEST YOURSELF

1. Answer the following objective questions : $3 \times 10 = 30$

(i) A body of mass m_1 strikes a stationary body of mass m_2. If the collision is elastic, the fraction of kinetic energy transmitted by the first body to the second body is

(a) $\frac{m_1 m_2}{m_1+m_2}$, (b) $\frac{2m_1 m_2}{(m_1+m_2)}$, (c) $\frac{4m_1 m_2}{(m_1+m_2)^2}$, (d) $\frac{2m_1 m_2}{(m_1+m_2)^2}$.

(ii) A body dropped from a height 1 m onto a floor rises to a height of 25 cm after the first rebound. The coefficient of restitution is (a) $\frac{3}{4}$, (b) $\frac{1}{4}$, (c) $\frac{1}{2}$, (d) $\frac{2}{8}$.

(iii) A horse pulls a wagon with a force of 360 N at an angle of 60° with horizontal at a speed of 10 kph. The power of the horse is (a) 1000 W, (b) 2000 W, (c) 500 W, (d) 750 W.

(iv) A running man has half the kinetic energy of a boy of half his mass. The man speeds up by 1 m/s and then has the same kinetic energy as the boy. The original speed of the boy was (a) $\sqrt{2}$ m/s, (b) $(\sqrt{2}-1)$ m/s, (c) 2 m/s, (d) $\sqrt{2}+1$ m/s.

(v) A boy whose mass is 30 kg climbs, with a constant speed, a vertical rope 6 m long in 10 s. The power of the boy during the climb ($g=10$ m s^{-2}) is

(a) 60 watt, (b) 3000 watt, (c) 180 watt, (d) 5 watt.

(vi) A ball approaches a moving wall of infinite mass with speed v along the normal to the wall. The speed of the wall is u away from the ball and $u<v$. The speed of the ball after 'elastic' collision is (a) $u+v$ away from the wall, (b) $2u+v$ away from the wall, (c) $v-u$ towards the wall, (d) $v-2u$ away from the wall.

(vii) A 30-gm bullet initially travelling horizontally at 50 m/s penetrates 10 cm into a wooden block. The average force it exerts is (a) 4000 N, (b) 375 N, (c) 2500 N, (d) 4500 N.

(viii) A force of 8 kgf raises a load of 4 kg through 5 m first and then the same force raises another load of 6 kg through the same distance. (a) It does greater work in raising 6 kg-load, (b) it does less work in raising 6 kg-load, (c) it does the same work in raising both the loads, (d) it cannot be said unless time in which work is done is given.

(ix) The power of an agent is given by

(a) $\vec{F} \cdot \vec{v}$, (b) $\vec{v} \cdot \vec{F}$, (c) $\vec{F} \times \vec{v}$, (d) $\vec{v} \times \vec{F}$.

(x) Which one of the following is 'watt' ?

(a) kg m s^{-2}, (b) N m s^{-2}, (c) kg $m^2 s^{-3}$, (d) N m s^{-1}.

Answer any five of the following : $5 \times 10 = 50$

2. Two perfectly smooth elastic discs *A* and *B*, one *k* times as massive as the other, rest on a smooth horizontal table. The disc *A* is made to move towards *B* with velocity *u* and make a head-on collision. Calculate the fraction of the kinetic energy transferred to *B* from that of *A*. Also show that the value of this fraction is the same whether *B* is *k* times as massive as *A* or vice versa.

3. A body of mass 10 kg and velocity 9 m/s strikes another body of mass 20 kg and velocity 6 m/s moving in the same direction. If the velocity of the first body becomes 6 m/s, find the coefficient of restitution and the loss in kinetic energy. What happens to this energy ?

4. A ball falls under gravity from a height of 10 m, with an initial velocity v_0. It collides with the ground, loses 50 per cent of its energy in collision and then rises to the same height. Find

(a) the initial velocity v_0 and

(b) the height to which the ball would rise, after collision, if the initial velocity v_0 was directed upward instead of downward.

(I.I.T. 1979)

5. A bullet weighing 50 gm is fired horizontally into wooden block 5 kg at rest on a horizontal surface. The coefficient of kinetic friction between the block and surface is 0·2. The bullet comes to rest in the block which moves 5 m. Find the speed of the bullet.

6. *O* is a point at the bottom of rough plane inclined at an angle α to the horizontal. Up to a vertical height *h* above *O* the coefficient of friction is 2tanα, above this height it is ½tanα. A particle is projected up the plane with velocity *V*. Prove that if $V^2 < 12gh$, the particle does not return to *O*.

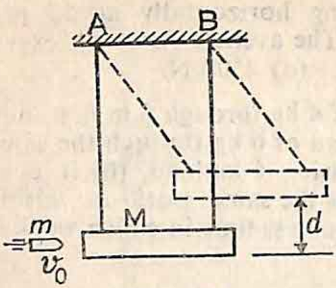


Fig. 4.18

7. A ballistic pendulum is used to measure bullet speeds. It consists of a wooden block of mass M suspended by two cords. Show that the speed of bullets can be estimated from the masses of the block and bullet and the distance 'd' by which the block rises parallel to itsef.

Answers of Exercises

1. 1·875. 2. ·1 m. 3. (a) −5·28 J; (b) ·24.
4. 27°, cannot be determined. 5. 10·6 m/s; 63°34′.
7. 64·8 cm/sec, 145·8 cm/sec.
8. C moves with the same speed along BG, that is, opposite to that of B.
9. $\frac{1}{8}ma^4t^2$. 10. $\frac{2m_1}{m_2+m_1}$; $\frac{4m_1m_2}{(m_1+m_2)^2}$. 11. 1 km/s.
12. $\sin\theta_{max}=\frac{m_2}{m_1}$. 13. 5 m/s. 14. $\frac{980}{8}$ cm/sec; $\theta=19°$.
15. $F=m_2g+m_1g$; m_1g.
16. $\frac{mM(u^2+V^2)}{(M+m)(mV^2+Mv^2)}$. 18. $h=\frac{m^2v_0^2\sin\alpha}{2M^2g(\sin\alpha+\mu\cos\alpha)}$.
19. $h_1=\frac{1}{18g}[v_0^2+2gl(1-\cos\alpha)]$,

$h_2=\frac{4}{18g}[v_0^2+2gl(1-\cos\alpha)]$.

20. 496 m/s. 21. 1·2 kg. 22. 308 m/s. 23. 3 s.
24. 10·5 m/s. 25. ·58 s. 27. $\frac{12}{18}d$.
28. $\sqrt{\frac{2m^2gh\cos^2\alpha}{(M+m)(M+m\sin^2\alpha)}}$. 29. 2 Ns.
30. $2m\sin\alpha/2\ \sqrt{\frac{gl}{(M+m)M}}$. 31. $\sqrt{\frac{g}{2l}}.x$; $N=\frac{Mg}{2\sqrt{2l^2}}x(2l-x)$.
32. $v=\sqrt{\frac{2mgh}{M+m\tan^2\alpha}}$.

33. (a) $h_1=\frac{1}{18g}[v_0{}^2+4gl\sin^2\alpha/2]$,

$$h_2=\frac{2}{9g}[v_0{}^2+4gl\sin^2\alpha/2];$$

(b) $h_3=\frac{1}{18g}[v_0{}^2+4gl\sin^2\alpha/2]$

34. $\frac{mg}{k}\left(1+\sqrt{1+\frac{2kl}{mg}}\right)$.

Answers of Test Questions

1. (i) (c), (ii) (c), (iii) (c), (iv) (d), (v) (c), (vi) (d), (vii) (b), (viii) (c), (ix) (a)+(b), (x) (d).

3. 22·5 Joule. 4. (a) 14 m/s, (b) 10 m.

5. 447 m/s. 7. $v_0=\frac{m+M}{m}\sqrt{2gd}$.

Solutions to Exercises

1. Work done by the frictional force

$$=-(\mu\times2\times g)\times{\cdot}6=-1{\cdot}29\mu.$$

Work done by the spring $=\frac{1}{2}kx^2=\frac{1}{2}\times1960\times{\cdot}15^2$.

$\therefore$ Net work done by external forces $=-1{\cdot}2g\mu+\frac{1}{2}\times1960\times{\cdot}15^2$.

By W-E Theorem,

$$-1{\cdot}2g\mu+\tfrac{1}{2}\times1960\times{\cdot}15^2=0$$

(since both initial and final velocities are zero)

$$\therefore\quad \mu=\frac{980\times{\cdot}15^2}{1{\cdot}2\times9{\cdot}8}=1{\cdot}875.\quad \textit{Ans.}$$

2. Let the spring be compressed by x m.

Work done by gravitational pull $=2g({\cdot}4+x)$.

Work done by the spring force $=-\frac{1}{2}kx^2=-\frac{1}{2}\times1960x^2=-980x^2$.

$\therefore$ Net work done by external forces $=2g({\cdot}4+x)-980x^2$.

By 'work-energy' theorem,

$$2g({\cdot}4+x)-980x^2=0$$

or

$$980x^2-2\times9{\cdot}8x-2\times9{\cdot}8\times{\cdot}4=0$$

$$100x^2-2x-{\cdot}8=0.$$

Hence $x={\cdot}1$ m or $-{\cdot}08$ m.

The negative value has no meaning.

$$\therefore\quad x={\cdot}1\text{ m}.\quad \textit{Ans.}$$

3. (b) Work done by gravitational pull $= mgh = 1 \times g \times 1{\cdot}2$
Work done by frictional force $= W$ (say)
Net work done $= 1{\cdot}2g + W$.
By 'work-energy' theorem,

$$1{\cdot}2g + W = \tfrac{1}{2} \times 1 \times 3{\cdot}6^2 - 0$$

or $W = \frac{1}{2} \times 3{\cdot}6^2 - 1{\cdot}2g = 6{\cdot}48 - 11{\cdot}76 = -5{\cdot}28$ J. *Ans.*

(a) Work done by frictional force $= -\mu \times (1 \times g) \times 2{\cdot}7$
Work done by the gravitational pull $= 0$.
By 'work-energy' theorem,

$$-\mu \times g + 2{\cdot}7 = 0 - \tfrac{1}{2} \times 1 \times 3{\cdot}6^2$$

or

$$\mu = \frac{1{\cdot}8 \times 3{\cdot}6}{9{\cdot}8 \times 2{\cdot}7} = {\cdot}24. \quad \textit{Ans.}$$

4. Let us take the direction of approach as x-axis.
Let A make angle θ with the x-axis on the negative side.
Considering conservation of momentum along x-axis and y-axis we have (taking m = mass of A and M = mass of B)

$$Mv + 0 = 0 + mu\cos\theta \text{ where } u = \text{speed of } A.$$

$$0 = M\frac{v}{2} - mu\sin\theta.$$

$$\therefore \quad Mv = mu\cos\theta$$

$$Mv = 2mu\sin\theta$$

$$\therefore \quad 2mu\sin\theta = mu\cos\theta$$

or

$$\tan\theta = \tfrac{1}{2} \text{ or } \theta = 27^\circ. \quad \textit{Ans.}$$

Using the identity $\cos^2\theta + \sin^2\theta = 1$ we have

$$\frac{M^2v^2}{m^2u^2} + \frac{M^2v^2}{4m^2u^2} = 1$$

$$\tfrac{5}{4}M^2v^2 = m^2u^2$$

or

$$u = \sqrt{\frac{5}{4}}\,\frac{M}{m}.$$

Since M and m are not known, u cannot be determined.

5. Let us consider eastward as x-axis and northward as y-axis. Let the combination move with speed u at an angle θ with the lx-axis.

Considering conservation of momentum along x and y-axis

$$900 \times \tfrac{100}{9} + 0 = (1200 + 900) \times u\cos\theta$$

$$0 + 1200 \times \tfrac{100}{8} = (1200 + 900) \times u\sin\theta$$

or

$$100 = 21u\cos\theta$$

$$200 = 21u\sin\theta$$

Squaring and adding

$$21^2u^2 = 50000$$

$$u = \tfrac{100}{21}\sqrt{5} = 13{\cdot}6 \text{ m/s.}$$

Dividing,

$$\tan\theta = 2 \text{ or } \theta = 63^\circ 24'.$$

Thus the interlocked vehicles will move with speed 10·6 m/s in direction 63°34′ north of east. *Ans.*

6. Let v be the velocity before hitting the ground.

Then $$v=\sqrt{2gh}.$$

Let v' be the velocity after the impact.

By Newton's law of collision of bodies

$$v'=e.v.$$

Let it go up by h'. Then $v'^2=2gh'$

or $$e^2v^2=2gh'.$$

Similarly, velocity after second rebound is $v''=ev'=e^2v$

and $$v''^2=2gh''$$

or $$e^4v^2=2gh''$$

H (total distance covered) $=h+2h'+2h''+2h'''+\ldots$

$$=h+2(h'+h''+h'''+\ldots\ldots)$$

$$=h+2\left(\frac{e^2v^2}{2g}+\frac{e^4v^2}{2g}+\frac{e^6v^2}{2g}+\ldots\right)$$

$$=h+\frac{2e^2v^2}{2g}(1+e^2+e^4+\ldots)$$

$$=h+\frac{e^2v^2}{g}\cdot\frac{1}{1-e^2},$$ (apply the rule of sum of a geometric series)

or $$H=h+\frac{e^2}{g(1-e^2)}.2gh \quad (\because \; v^2=2gh)$$

$$=h+\frac{2e^2h}{1-e^2}$$

or $$H=h\times\frac{1+e^2}{1-e^2}.$$ *Proved.*

The time of the first fall is $v=gt$ or $t=\sqrt{\frac{2h}{g}}.$

The time of first ascent or descent is $v'=gt'$

or $$t'=\frac{ev}{g}=e\sqrt{\frac{2h}{g}}.$$

The time of the second ascent or descent is

$$v'=gt'' \quad \text{or} \quad t''=ev'=e^2v=e^2\sqrt{\frac{2h}{g}}.$$

$$\therefore \text{ Total time} = t+2t'+2t''+2t'''+\ldots$$

$$=\sqrt{\frac{2h}{g}}+2e\sqrt{\frac{2h}{g}}+2e^2\sqrt{\frac{2h}{g}}+2e^3\sqrt{\frac{2h}{g}}+\ldots$$

$$=\sqrt{\frac{2h}{g}}+2e\sqrt{\frac{2h}{g}}(1+e+e^2+\ldots)$$

$$=\sqrt{\frac{2h}{g}}+2e\sqrt{\frac{2h}{g}}\cdot\frac{1}{1-e}$$

$$=\sqrt{\frac{2h}{g}}\times\frac{1+e}{1-e}.\quad \textit{Proved.}$$

7. Since radius is double, mass is 8 times greater. Hence if mass of the smaller ball is m, the mass of the bigger ball is $8m$. Considering conservation of momentum,

$$8m\times 81 = 8mu_1+mu_2$$

or

$$8(81-u_1)=u_2. \qquad \ldots (i)$$

From conservation of kinetic energy

$$\tfrac{1}{2}.8m81^2=\tfrac{1}{2}.8mu_1^2+\tfrac{1}{2}mu_2^2$$

or

$$8\times 81^2=8u_1^2+u_2^2$$

or

$$8(81^2-u_1^2)=u_2^2$$

$$8(81-u_1)(81+u_1)=u_2^2. \qquad \ldots (ii)$$

Combining (i) and (ii),

$$81+u_1=u_2. \qquad \ldots (iii)$$

Hence

$$u_1=64{\cdot}8 \text{ and } u_2=145{\cdot}8.$$

Thus after collision the bigger ball will move with speed 64·8 cm/sec and the smaller ball will move with speed 145·8 cm/sec. *Ans.*

8. Let us consider conservation of momentum along AG and perpendicular to it. Let C return with speed u at an angle θ with AG (anticlockwise).

$$mV-mV\cos60^\circ-mV\cos60^\circ=0+mV\cos60^\circ+mu\cos\theta$$

or

$$0=\tfrac{1}{2}V+u\cos\theta$$

or

$$-\tfrac{1}{2}V=u\cos\theta \qquad \ldots (i)$$

$$0=mV\sin60^\circ-mu\sin\theta$$

or

$$V\sin60^\circ=u\sin\theta$$

$$\frac{\sqrt{3}}{2}V=u\sin\theta \qquad \ldots (ii)$$

Squaring and adding (i) and (ii),

$$\frac{V^2}{4}+\frac{3V^2}{4}=u^2$$

or

$$u=V$$

Dividing,

$$\tan\theta=-\sqrt{3}$$

$$\theta=120^\circ.$$

Thus C moves with the same speed along BG, that is, opposite to that of B. *Ans.*

9. According to the 'work-energy' theorem 'work done by all forces is equal to change in kinetic energy'.

$$\therefore \quad W = K_f - K_i = K_f \qquad (\because \quad K_i = 0).$$

Now $$v = a\sqrt{5} \quad \text{or} \quad v^2 = a^2 s$$

Differentiating, $$2v\frac{dv}{dt} = a^2\frac{ds}{dt}$$

$$= a^2 v \qquad \left(\because \quad \frac{ds}{dt} = v\right)$$

or $$2dv = a^2 dt.$$

Integrating, $2v = a^2 t +$ a constant.

At the moment $t = 0$, $v = 0$;

$\therefore$ the constant $= 0$

$$\therefore \quad v = \tfrac{1}{2}a^2 t.$$

$$\therefore \quad W = K_f = \tfrac{1}{2}mv^2 = \tfrac{1}{2}.m(\tfrac{1}{2}a^2 t)^2 = \tfrac{1}{8}ma^4 t^2. \quad \textit{Ans.}$$

10. Let u_1 be the velocity of the colliding particle before collision. Let v_1 and v_2 be the velocities of m_1 and m_2 after collision. Let us consider the line of approach as x-axis. Let m_2 move at angle θ with x-axis (on the other side).

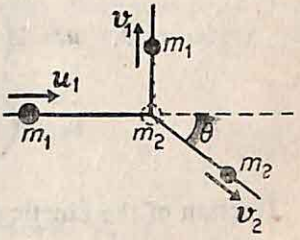


Fig. 4.19

Considering conservation of momentum along x-axis and perpendicular to it,

$$m_1 u_1 + 0 = 0 + m_2 v_2 \cos\theta$$

or $$m_1 u_1 = m_2 v_2 \cos\theta \qquad \text{.. (i)}$$

$$0 = m_1 v_1 - m_2 v_2 \sin\theta$$

or $$m_1 v_1 = m_2 v_2 \sin\theta. \qquad \text{.. ii)}$$

Since collision is elastic, kinetic energy is conserved.

$$\therefore \quad \tfrac{1}{2}m_1 u_1^2 = \tfrac{1}{2}m_1 v_1^2 + \tfrac{1}{2}m_2 v_2^2$$

$$m_1(u_1^2 - v_1^2) = m_2 v_2^2$$

or

$$v_2^2 = \frac{m_1}{m_2}(u_1^2 - v_1^2) \qquad \text{.. (ii)}$$

Squaring and adding (i) and (ii),

$$m_1^2(u_1^2 + v_1^2) = m_2^2 v_2$$

$$\therefore \quad m_1^2(u_1^2 + v_1^2) = m_2^2.\frac{m_1}{m_2}(u_1^2 - v_1^2) = m_1 m_2(u_1^2 - v_1^2)$$

or $$(m_1^2 + m_1 m_2)v_1^2 = (m_1 m_2 - m_1^2)u_1^2$$

or $$m_1(m_1 + m_2)v_1^2 = m_1(m_2 - m_1)u_1^2$$

or

$$\frac{v_1^2}{u_1^2}=\frac{m_2-m_1}{m_2+m_1}.$$

Energy loss by the striking particle $=\frac{1}{2}m_1u_1^2-\frac{1}{2}m_1v_1^2$

$\therefore$ fraction of the kinetic energy loss $=\dfrac{\frac{1}{2}m_1u_1^2-\frac{1}{2}m_1v^2}{\frac{1}{2}m_1u_1^2}$

$$=1-\frac{v_1^2}{u_1^2}=\left(1-\frac{m_2-m_1}{m_2+m_1}\right)=\frac{2m_1}{m_2+m_1}.\quad \textit{Ans.}$$

(ii) In head-on collision,

$$m_1u_1=m_1v_1+m_2v_2$$

or

$$m_1(u_1-v_1)=m_2v_2 \qquad \text{.. (i)}$$

By kinetic energy conservation,

$$\tfrac{1}{2}m_1u_1^2=\tfrac{1}{2}m_1v^2+\tfrac{1}{2}m_2v_2^2$$

or

$$m_1(u_1^2-v_1^2)=m_2v_2^2$$

Dividing (ii) by (i),

$$u_1+v_1=v_2$$

and

$$u_1-v_1=\frac{m_2}{m_1}v_2$$

$$\therefore\quad u_1=\tfrac{1}{2}\left(1+\frac{m_2}{m_1}\right)v_2=\tfrac{1}{2}\frac{m_1+m_2}{m_1}v_2$$

$$v_1=\tfrac{1}{2}\left(1-\frac{m_2}{m_1}\right)v_2=\tfrac{1}{2}\frac{m_1-m_2}{m_1}v_2$$

fraction of the kinetic energy loss $=\left(1-\dfrac{v_1^2}{u_1^2}\right)=\left[1-\left(\dfrac{m_1-m_2}{m_1+m_2}\right)^2\right]$

$$=\frac{4m_1m_2}{(m_1+m_2)^2}.\quad \textit{Ans.}$$

11. From conservation of momentum

$$mv=\frac{m}{3}v_1+\frac{m}{3}v_2+\frac{m}{3}v_3$$

or

$$3v=v_1+v_2+v_3 \qquad \text{.. (i)}$$

Now, $n=\dfrac{\frac{1}{2}m/3v_1^2+\frac{1}{2}m/3v_2^2+\frac{1}{2}m/3v_3^2}{\frac{1}{2}mv^2}$. (This is given).

$$\therefore\quad 3nv^2=v_1^2+v_2^2+v_3^2. \qquad \text{.. (ii)}$$

Eliminating v_3 between (i) and (ii),

$$3nv^2=v_1^2+v_2^2+(3v-v_1-v^2)^2$$

$$=2v_1^2+2v_2^2+9v^2-6vv_1+2v_1v_2-6vv_2.$$

Arranging in quadratic in v_1,

$$2v_1^2-2v_1(3v-v_2)+[(9-3n)v^2+2v_2^2-6vv_2]=0.$$

Since v_1 is real, the discriminant must be greater or equal to zero.

$$4(3v-v_2)^2-4\times 2[(9-3n)v^2+2v_2^2+6vv_2]\geqslant 0$$

or $$9v^2-6vv_2+v_2^2-2(9-3n)v^2-4v_2^2+12vv_2\geqslant 0$$

or $$9v^2+6vv_2-3v_2^2-18v^2+6nv^2\geqslant 0$$

or $$-9v^2+6vv_2-3v_2^2+6nv^2\geqslant 0$$

or $$6nv^2\geqslant 3v_2^2-6vv_2+9v^2$$

or $$2nv^2\geqslant v_2^2-2vv_2+3v^2$$

or $$2nv^2\geqslant(v_2-v)^2+2v^2$$

or $$(2n-2)v^2\geqslant(v_2-v)^2$$

or $$\sqrt{(2n-2)}v\geqslant v_2-v$$

or $$v_2\leqslant v(1+\sqrt{2n-2}).$$

$$\therefore\quad v_2\text{ (maximum)}=v(1+\sqrt{2n-2})$$

$$=500(1+\sqrt{2\times 1{\cdot}5-2})=1000\text{ m/s}=1\text{ km/s}.\quad \textit{Ans.}$$

12. Let the striking particle approach the stationary particle from the left and take this direction as x-axis. Let u_1 be the velocity of approach of m_1. Further let v_1 and v_2 be the velocities of m_1 and m_2 at angles θ and θ' in anticlockwise and clockwise with x-axis respectively. Considering conservation of momentum along x-axis and y-axis,

$$m_1u_1=m_1v_1\cos\theta+m_2v_2\cos\theta'$$

or $$(m_1u_1-m_1v_1\cos\theta)=m_2v_2\cos\theta' \quad\text{.. (i)}$$

and $$0=m_1v_1\sin\theta-m_2v_2\sin\theta'$$

or $$m_1v_1\sin\theta=m_2v_2\sin\theta' \quad\text{.. (ii)}$$

Squaring and adding (i) and (ii) to eliminate θ',

$$(m_1u_1-m_1v_1\cos\theta)^2+m_1^2v_1^2\sin^2\theta=m_2^2v_2^2$$

or $$m_1^2u_1^2-2m_1^2u_1v_1\cos\theta+m_1^2v_1^2=m_2^2v_2^2 \quad\text{.. (iii)}$$

From conservation of kinetic energy as the collision is elastic we have

$$\tfrac{1}{2}m_1u_1^2=\tfrac{1}{2}m_1v_1^2+\tfrac{1}{2}m_2v_2^2$$

or $$m_1(u_1^2-v_1^2)=m_2v_2^2 \quad\text{.. (iv).}$$

From (iii) and (iv) we have

$$m_1^2u_1^2-2m_1^2u_1v_1\cos\theta+m_1^2v_1^2=m_1m_2(u_1^2-v_1^2).$$

Arranging in quadratic of v_1 we have

$$m_1(m_1+m_2)v_1^2-v_1(2m_1^2u_1\cos\theta)+m_1cm_1-m_2)u_1^2=0.$$

Since v_1 is real,

$$4m_1^4u_1^2\cos^2\theta-4m_1(m_1+m_2)m_1(m_1-m_2)u_1^2\geqslant 0$$

or $$m_1^2\cos^2\theta-(m_1^2-m_2^2)\geqslant 0$$

or $$m_1^2-m_1^2\sin^2\theta-m_1^2+m_2^2\geqslant 0$$

or $$m_2^2\geqslant m_1^2\sin^2\theta$$

or $$\sin\theta \leqslant \frac{m_2}{m_1}$$

$$\therefore \quad \sin\theta_{max} = \frac{m_2}{m_1}. \quad \textit{Ans.}$$

13. We have, $$m_1 v = m_1 v_1 + m_2 v_2$$
—by conservation of momentum.

Fractional decrease in K.E. $= \dfrac{K_i - K_f}{K_i} = n$ (given)

or $$1 - n = \frac{\frac{1}{2} m_1 {v_1}^2 + \frac{1}{2} m_2 {v_2}^2}{\frac{1}{2} m_1 v^2}$$

or $$m_1(1-n)v^2 = m_1 v^2 + m_2 {v_2}^2$$

$$\therefore \quad m_1(1-n)v^2 = m_1 {v_1}^2 + m_2 \left(\frac{m_1 v - m_1 v_1}{m_2} \right)^2$$

$$(1-n)v^2 = {v_1}^2 + \frac{m_1}{m_2}(v - v_1)^2.$$

Putting $m_1 = m_2$ as they are of the same mass,

$$(1-n)v^2 = {v_1}^2 + (v - v_1)^2$$

or $$2{v_1}^2 - 2v_1 v + nv^2 = 0$$

or $$v_1 = \frac{2v \pm \sqrt{4v^2 - 4.2.nv^2}}{4}$$

$$= \tfrac{1}{2}(v \pm \sqrt{v^2 - 2nv^2}) = \tfrac{1}{2}v(1 \pm \sqrt{1 - 2n}).$$

The positive sign is not admissible as v_1 cannot be greater than v.

$$\therefore \quad v_1 = \tfrac{1}{2}v(1 - \sqrt{1-2n})$$

$$= \tfrac{1}{2} \times 10(1 - \sqrt{1 - 2 \times {\cdot}01}) = 5 \text{ m/s.} \quad \textit{Ans.}$$

14. Let v be the velocity acquired by the ball on reaching the lowest point. Then

$$\tfrac{1}{2} mv^2 = mg(l - l\cos\theta)$$

$$v = \sqrt{2gl(1-\cos\theta)} = \sqrt{2 \times 980 \times 980(1 - \cos 60^\circ)} = 980 \text{ cm/sec.}$$

By conservation of momentum,

$$10 \times 980 = 10v_1 + 20v_2$$

or $$10(980 - v_1) = 20v_2 \qquad \text{.. (i)}$$

By conservation of kinetic energy,

$$\tfrac{1}{2} \times 10 \times 980^2 = \tfrac{1}{2} \times 10{v_1}^2 + \tfrac{1}{2} \times 20 \times {v_2}^2$$

$$10(980^2 - {v_1}^2) = 20{v_2}^2$$

or $$10(980 - v_1)(980 + v_1) = 20{v_2}^2 \qquad \text{.. (ii)}$$

Dividing (ii) by (i),

$$980 + v_1 = v_2$$

$$980 - v_1 = 2v_2.$$

Hence $v_1 = -\frac{980}{3}$ cm/sec. *Ans.*

The negative sign shows that it will be returned along its path after collision.

The angular displacement is θ (say). Then

$$\tfrac{1}{2}m(\tfrac{980}{3})^2 = m \times 980 \times 980(1-\cos\theta)$$

or $$\tfrac{1}{18} = 1-\cos\theta, \text{ or } \cos\vartheta = \tfrac{17}{18} = \cdot 944$$

or $$\theta = 19°. \quad \textit{Ans.}$$

15. Let the reference line be the horizontal line through the c.g. of the lower mass m_1. Let l be the natural length of the spring and ϵ is the initial compression needed.

Then initial energy of the system $= \frac{1}{2}k\epsilon^2 + m_2g(l-\epsilon)$
(refer to figure of example 8).

Let the spring extend by x after the force is released.

The final energy of the system $= \frac{1}{2}kx^2 + mg(l+x)$.

By energy conservation principle,

$$\tfrac{1}{2}k\epsilon^2 + m_2g(l-\epsilon) = \tfrac{1}{2}kx^2 + m_2g(l+x)$$

or $$kx^2 + 2m_2gx - k\epsilon^2 + 2m_2g\epsilon = 0$$

or $$x = \frac{-2m_2g \pm \sqrt{4m_2{}^2g^2 - 4k(2m_2g\epsilon - k\epsilon^2)}}{2k}$$

or $$kx = -m_2g \pm \sqrt{(m_2g - k\epsilon)^2}$$
$$= -k\epsilon \text{ or } k\epsilon - 2m_2g$$

$-k\epsilon$ is not acceptable.

$$\therefore \quad kx = k\epsilon - 2m_2g.$$

The lower one will bounce up if $kx \geqslant m_1g$

or $$k\epsilon - 2m_2g \geqslant m_1g$$

or $k\epsilon \geqslant 2m_2g + m_1g$, or $k\epsilon = 2m_2g + m_1g$ for just lifting.

Let F be the force needed to compress the spring by ϵ. The weight m_2g also compresses the spring.

$$\therefore \quad F + m_2g = k\epsilon$$
$$\therefore \quad F + m_2g = 2m_2g + m_1g$$

or $$F = m_2g + m_1g. \quad \textit{Ans.}$$

When $m_2 = 0$, $F = m_1g$. *Ans.*

16. Considering conservation of momentum along x- and y-axis,

$$mV + 0 = (m+M)u\cos\theta$$

where u = velocity of the interlocked body and θ is the direction with x-axis

$$0 + Mv = (m+M)u\sin\vartheta.$$

Dividing, we have

$$\tan\theta = \frac{M}{m} \cdot \frac{v}{V}.$$

Squaring and adding,

$$m^2V^2+M^2v^2=(m+M)^2u^2$$

or

$$u=\frac{\sqrt{m^2V^2+M^2v^2}}{m+M}.$$

So the interlocked body moves with speed $\dfrac{\sqrt{m^2V^2+M^2v^2}}{M+m}$ in direction $\tan^{-1}\dfrac{Mv}{mv}$ with the x-axis.

The fraction of the initial kinetic energy converted into heat

$$=\frac{K_i-K_f}{K_f}=1-\frac{K_f}{K_i}=1-\frac{\frac{1}{2}(m+M)u^2}{\frac{1}{2}mV^2+\frac{1}{2}Mv^2}$$

$$=1-\frac{(m+M)u^2}{mV^2+Mv^2}=1-\frac{(m+M)\cdot\dfrac{m^2V^2+M^2v^2}{(M+m)^2}}{mV^2+Mv^2}$$

$$=1-\frac{m^2V^2+M^2v^2}{(mV^2+Mv^2)(M+m)}$$

$$=\frac{mM(v^2+V^2)}{(M+m)(mV^2+Mv^2)}.\quad \textit{Ans.}$$

17. Let α and β be the angles made by the striking ball and the stationary ball after collision with the x-axis (i.e. line of approach) in the anticlockwise and clockwise sense.

From conservation of momentum,

$$m\times 50=m\times 30\cos\alpha+m\times v\cos\beta$$

$$50=30\cos\alpha+v\cos\beta \quad \text{.. (i)}$$

$$0=m\times 30\sin\alpha-m\times v\sin\beta$$

or

$$30\sin\alpha=v\sin\beta \quad \text{.. (ii).}$$

From conservation of kinetic energy,

$$\tfrac{1}{2}m\times 50^2=\tfrac{1}{2}\times m\times 30^2+\tfrac{1}{2}m\times v^2$$

$$v=40 \text{ cm/sec.} \quad \textit{Ans.}$$

$$\therefore\quad 50=30\cos\alpha+40\cos\beta$$

or

$$5=3\cos\alpha+4\cos\beta \quad \text{.. (iii)}$$

and

$$3\sin\alpha=4\sin\beta \quad \text{.. (iv)}$$

or

$$0=4\sin\beta-3\sin\alpha.$$

Squaring and adding (iii) and (iv),

$$25=9\cos^2\alpha+24\cos\alpha\cos\beta+16\cos^2\beta+16\sin^2\beta-24\sin\beta\sin\alpha+9\sin^2\alpha$$

or

$$25=9+16+24(\cos\alpha\cos\beta-\sin\alpha\sin\beta)$$

or

$$0=24\cos(\alpha+\beta)$$

or

$$\cos(\alpha+\beta)=0$$

or $$\alpha+\beta=\frac{\pi}{2}.\quad \textit{Proved.}$$

18. By conservation of momentum,

$$mv_0=MV \text{ where } V \text{ is the velocity of recoil}$$

or $$V=\frac{mv_0}{M}.$$

Let the cannon go up the plane by vertical height h.

Work done by gravity $=-Mgh$.

Work done by friction $=-\mu Mg\cos\alpha\times\frac{h}{\sin\alpha}$.

Work done by reaction of the plane $=0$.

$\therefore$ Net work done by external forces $=-Mgh-\mu Mgh\cot\alpha$.

By 'work-energy theorem',

$$-Mgh(1+\mu\cot\alpha)=0-\tfrac{1}{2}.M\frac{m^2v_0^2}{M^2}$$

$$Mgh(1+\mu\cot\alpha)=\frac{m^2v_0^2}{2M}$$

$$h=\frac{m^2v_0^2\sin\alpha}{2M^2g(\sin\alpha+\mu\cos\alpha)}.\quad \textit{Ans.}$$

19. The velocity acquired by m on reaching the lowest position is v (say).

$$\tfrac{1}{2}mv_0^2+mgl(1-\cos\alpha)=\tfrac{1}{2}v^2$$

or $$v^2=v_0^2+2gl(1-\cos\alpha).$$

(i) By momentum conservation,

$$mv=mv_1+2mv_2$$

or $$v=v_1+2v_2,\quad \text{or}\quad (v-v_1)=2v_2.$$

By 'kinetic energy conservation',

$$\tfrac{1}{2}mv^2=\tfrac{1}{2}mv_1^2+\tfrac{1}{2}(2m)v_2^2$$

or $$v^2=v_1^2+2v_2^2$$

or $$(v^2-v_1^2)=2v_2^2,\ \text{or}\ (v-v_1)(v_1+v)=2v_2^2$$

or $$v_1+v=v_2$$

$$\therefore\quad v_2=\tfrac{2}{3}v\quad\text{and}\quad v_1=-\frac{v}{3}.$$

Let m rise by h_1 and $2m$ by h_2. Then

$$\tfrac{1}{2}mv_1^2=mgh,\quad \text{or } gh_1=\tfrac{1}{2}\cdot\frac{v^2}{9}=\tfrac{1}{18}[v_0^2+2gl(1-\cos\alpha)]$$

$$h_1=\frac{1}{18g}[v_0^2+2gl(1-\cos\alpha)].\quad \textit{Ans.}$$

$$\tfrac{1}{2}2mv_2^2 = 2mgh_2 \text{ or } gh_2 = \tfrac{1}{2}.\frac{4v^2}{9} = \tfrac{4}{18}[v_0^2 + 2gl(1-\cos\alpha)]$$

or $$h_2 = \frac{4}{18g}[v_0^2 + 2gl(1-\cos\alpha)]. \quad \textit{Ans.}$$

20. By 'conservation of momentum',

$$(10\times 10^{-3})v = (5+ 01)v', \text{ or } v' = \frac{10^{-2}v}{5{\cdot}01} = \frac{v}{501} \text{ m/s.}$$

By 'energy conservation',

$$\tfrac{1}{2}\times 5\ 01 \times \left(\frac{v}{501}\right)^2 = 5{\cdot}01\times 9{\cdot}8\times(5\times 10^{-2})$$

$$\left(\frac{v}{501}\right)^2 = 2\times 9{\cdot}8\times 5\times 10^{-2}$$

or $$v = 501\sqrt{\cdot 98} = 496 \text{ m/s.} \quad \textit{Ans.}$$

21. 'By conservation of momentum',

$$2\times v = 2\times\frac{v}{4} + mv'$$

or $$2\times\frac{3v}{4} = mv', \text{ or } 3v = 2mv' \quad .. \quad \text{(i).}$$

By 'conservation of energy',

$$\tfrac{1}{2}\times 2\times v^2 = \tfrac{1}{2}\times 2\times\frac{v^2}{16} + \tfrac{1}{2}mv'^2$$

$$2v^2 = \frac{v^2}{8} + mv'^2$$

or $$\frac{15v^2}{8} = mv'^2 = m.\frac{9v^2}{4m^2}$$

or $$m = \tfrac{6}{5} = 1{\cdot}2 \text{ kg.} \quad \textit{Ans.}$$

22. 'By conservation of momentum',

$$10\times 10^{-3}\times v = (2+{\cdot}01)v', \text{ or } v' = \frac{10^{-2}v}{2{\cdot}01} = \frac{v}{201}.$$

'By energy conservation',

$$\tfrac{1}{2}\times 2{\cdot}01\times\left(\frac{v}{201}\right)^2 = 2{\cdot}01\times 9{\cdot}8\times{\cdot}12$$

$$\therefore \quad v = 201\sqrt{9{\cdot}8\times{\cdot}12\times 2} = 308 \text{ m/s.} \quad \textit{Ans.}$$

23. Velocity acquired by the ball $= 9{\cdot}8 \times 2 = 19{\cdot}6$ m/s.

The ball is incident at an angle 60° with the plane. Let it rebound at angle α with the plane with speed v. Since the plane is smooth, there is no change in velocity along it.

$\therefore\ 19{\cdot}6\cos 60° = v\cos\alpha$

or $v\cos\alpha = 9{\cdot}8.$

By Newton's law of collision,

$v\sin\alpha = e \times 19{\cdot}6\sin 60°$

$= \frac{3}{4} \times 19{\cdot}6 \times {\cdot}866 = 12{\cdot}7.$

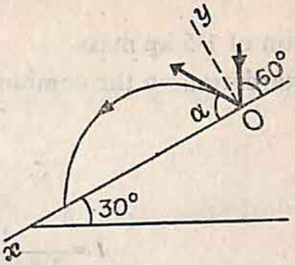


Fig. 4.20

The ball is projected after collision with speed v at an angle α with the plane The accelerations along and perpendicular (towards) to the plane are $g\sin 30°$ and $g\cos 30°$.

Taking reference of axes along and perpendicular to the plane we have

$$0 = v\sin\alpha t + \tfrac{1}{2}(-g\cos 30°)t^2$$

(from consideration of motion along y-axis)

or $$v\sin\alpha = \tfrac{1}{2}g\cos 30° t$$

or $$12{\cdot}7 = \tfrac{1}{2} \times 9{\cdot}8 \times {\cdot}866 t$$

or $$t = \frac{2 \times 12{\cdot}7}{9{\cdot}8 \times {\cdot}866} = 3s. \quad \textit{Ans.}$$

24. Let v be the velocity of the bullet.

By conservation of momentum,

$${\cdot}1 \times v = 1{\cdot}1 \times v' \quad \text{or } v' = \frac{v}{11}.$$

Mechanical energy of the system after collision

$$= \tfrac{1}{2} \times 1{\cdot}1 \times \left(\frac{v}{11}\right)^2.$$

This energy is used in compressing the spring.

Work done in compressing the spring $= \frac{1}{2}kx^2 = \frac{1}{2} \times 100 \times {\cdot}1^2.$

$$\therefore\ \tfrac{1}{2} \times 100 \times {\cdot}1^2 = \tfrac{1}{2} \times 1{\cdot}1 \times \left(\frac{v}{11}\right)^2$$

or $$\sqrt{\frac{100 \times {\cdot}01}{1{\cdot}1}} \times 11 = v \quad \text{or } v = 10{\cdot}5 \text{ m/s.} \quad \textit{Ans.}$$

25. By conservation of momentum (direction of 2 kg mass is x-axis),

$$2 \times 1 + 1{\cdot}5 \times (-2) = (2 + 1{\cdot}5)v$$

$$v = -\frac{1}{3{\cdot}5}.$$

Thus the combined mass will move with speed $\frac{1}{3{\cdot}5}$ m/s in the direction of 1·5 kg mass.

Retardation on the combined mass $= \mu g$

$$= {\cdot}05 \times 9{\cdot}8 = {\cdot}49 \text{ m/s}^2.$$

$$\therefore \quad \frac{1}{3{\cdot}5} = {\cdot}49t$$

or $$t = \frac{1}{3{\cdot}5 \times {\cdot}49} = {\cdot}58 \text{ s.} \quad \textit{Ans.}$$

26. Let u_1 and u_2 be the velocity of m and $2m$ in the direction of u.

By conservation of momentum,

$$3mu = mu_1 + 2mu_2$$

or $$3u = u_1 + 2u_2 \qquad \ldots \quad \text{(i)}.$$

Additional energy generated $= K_f - K_i = \frac{1}{2}mu_1^2 + \frac{1}{2}.2mu_2^2 - \frac{1}{2}.3mu^2$

$$= 12mu^2 \text{ (given)}$$

or $$u_1^2 + 2u_2^2 = 27u^2 \qquad \ldots \quad \text{(ii)}.$$

Putting the value of u_1 from (i) in (ii),

$$(3u - 2u_3)^2 + 2u_2^2 = 27u^2$$

or $$u_2^2 - 2uu_2 - 3u^2 = 0$$

or $$(u_2 - u)^2 = 4u^2 \quad \text{or} \quad u_2 = u \pm 2u = 3u \text{ or } -u.$$

Putting the values of u_1 in (i), $u_1 = -3u$ or $5u$.

The values $u_1 = -3u$ and $u_2 = +3u$ are not acceptable because the two masses being unequal cannot have equal values of velocities.

$$\therefore \quad u_1 = 5u \text{ and } u_2 = -u.$$

Relative velocity = vector difference of absolute velocities

$$= 5u - (-u) = 6\mu. \quad \textit{Proved.}$$

27. Let u be the velocity of A towards the wall and v_1 is its velocity away from the wall after collision and v_2 is the veloeity of B towards the wall.

By Newton's law of collision,

$$(v_1 + v_2) = eu.$$

By 'conservation of momentum towards the wall',

$$mu = -mv_1 + 2mv_2$$

or $$u = -v_1 + 2v_2 \quad \text{or} \quad 2v_2 - v_1 = u.$$

Solving for v_1 and v_2 we have, $v_2 = \frac{1}{3}(1+e)u$, $v_1 = \frac{(2e-1)}{3}u$.

The time of travel of B up to the wall is $\frac{d}{v_2}$.

By this time A will move away from the wall by

$$v_1\times\frac{d}{v_2}=d\times\frac{2e-1}{1+e}.$$

So the distance between A and B when B is just going to strike the wall is $\left(d\frac{2e-1}{1+e}+d\right)=\frac{3ed}{1+e}$.

The velocity of B after collision with the wall is ev_2 away from the wall.

Let them meet at a distance x from the wall. Then the time in which B travels x, in the same time A travels $\left(x-\frac{3ed}{1+e}\right)$.

$$\therefore\quad \frac{x-\frac{3ed}{1+e}}{v_1}=\frac{x}{ev_2}$$

or
$$x-\frac{3ed}{1+e}=\frac{x}{e}\cdot\frac{v_1}{v_2}=\frac{x}{e}\cdot\frac{2e-1}{1+e}$$

or
$$x\left[1-\frac{2e-1}{e(1+e)}\right]=\frac{3ed}{1+e}$$

or
$$x=\frac{3ed}{e^2-e+1}=\frac{3\times\frac{1}{4}d}{(\frac{1}{4})^2-\frac{1}{4}+1}=\frac{12}{13}d.\quad \textit{Ans.}$$

28. Let u be the relative velocity of m down the wedge. Then $u\cos\alpha$ is the relative rightward velocity of m and $u\sin\alpha$ is the relative downward vertical velocity relative to M.

Then $u\cos\alpha=u_x-(-v)$ where u_x is the rightward absolute velocity of m and v is the leftward absolute velocity of M

or
$$u\cos\alpha-v=u_x$$

and $u\sin\alpha=u_y$ where $u_y=$ absolute downward vertical velocity of m.

By 'momentum conservation to the right',

$$mu_x+(-Mv)=0\quad\text{or}\quad mu_x=Mv.$$

By 'energy conservation',

$$\tfrac{1}{2}Mv^2+\tfrac{1}{2}m(u_x{}^2+u_y{}^2)=mgh$$

$$Mv^2+m\left[\frac{M^2v^2}{m^2}+u_y{}^2\right]=2mgh.$$

Now $u_y=u\sin\alpha=\dfrac{(u_x+v)}{\cos\alpha}\sin\alpha=\left(\dfrac{M}{m}v+v\right)\tan\alpha=v\tan\alpha\left(1+\dfrac{M}{m}\right).$

$$\therefore \quad Mv^2+\frac{M^2v^2}{m}+m.v^2\tan^2\alpha\left(1+\frac{M}{m}\right)^2=2mgh$$

$$Mv^2\left(1+\frac{M}{m}\right)+mv^2\tan^2\alpha\left(1+\frac{M}{m}\right)^2=2mgh$$

$$\text{or} \quad v^2\left[\frac{M(M+m)}{m}+\frac{(M+m)^2\tan^2\alpha}{m}\right]=2mgh$$

$$\text{or} \quad v^2=\frac{2m^2gh}{M(M+m)+(M+m)^2+\tan^2\alpha}=\frac{2m^3gh\cos^2\alpha}{(M+m)(M+m\sin^2\alpha)}$$

$$v=\sqrt{\frac{2m^2gh\cos^2\alpha}{(M+m)(M+m\sin^2\alpha)}}. \quad \textit{Ans.}$$

29. Let v be the velocity on reaching the surface. Then

$$v=\sqrt{2gh}.$$

By Newton's law of collision, the velocity v' after the first impact is $v'=ev$ (upward). Similarly, $v''=ev'=e^2v$; $v'''=e^3v$; .....

$\therefore$ Momentum imparted to the slab $=mv-(-mv')$
$=m(v+v')=mv(1+e)$.

Similarly, momentum imparted to the slab in the second impact
$=m(v'+v'')=m(v'+ev')=m(1+e)v'=m(1+e)ev$;

momentum imparted to the slab in the 3rd impact
$=m(v''+v''')=m(v''+ev'')=m(1+e)v''=m(1+e)e^2v$
and so on.

$\therefore$ Total momentum imparted

$$=mv(1+e)+mve(1+e)+mve^2(1+e)+\dots$$

$$=mv(1+e)(1+e+e^2+\dots)$$

$$=mv(1+e)\times\frac{1}{1-e}$$

$$=mv\frac{1+e}{1-e}.=m\sqrt{2gh}\,\frac{1+e}{1-e}$$

$$=\cdot05\sqrt{2\times9\cdot8\times1}\,\frac{1+\cdot8}{1-\cdot8}=\frac{\cdot05\sqrt{19\cdot6}\times1\cdot8}{\cdot2}$$

$=2$ Ns. *Ans.*

30. Let v be the leftward velocity of the wagon (absolute, that is, relative to the earth). Let u be the velocity of the pendulum in a frame fixed to the wagon. Then $u\cos\beta$ is the relative horizontal velocity of the bob and $u\sin\beta$ is its vertical velocity. Let v_x and v_y be the absolute horizontal and vertical downward velocity of the bob.

Then $u\cos\beta$ = vector difference of absolute velocities

$$= v_x - (-v)$$

or $$v_x = u\cos\beta - v$$

or $$u\sin\beta = v_y.$$

There is no external force on the system in the horizontal direction.

$\therefore$ By 'momentum conservation to the right',

$$0 = m(u\cos\beta - v) - Mv$$

or $$u\cos\beta - v = \frac{M}{m}v \quad \text{or} \quad u = \frac{(M+m)v}{m\cos\beta}.$$

Kinetic energy of the bob $= \frac{1}{2}m(v_x^2 + v_y^2)$.

By conservation of energy,

$$mgl(1-\cos\alpha) = mgl(1-\cos\beta) + \tfrac{1}{2}Mv^2 + \tfrac{1}{2}m[(u\cos\beta - v)^2 + u^2\sin^2\beta]$$

or $$2mgl(\cos\beta - \cos\alpha) = Mv^2 + m.\frac{M^2v^2}{m^2} + m\sin^2\beta.\frac{(M+m)^2v^2}{m^2\cos^2\beta}$$

$$= Mv^2\left(1 + \frac{M}{m}\right) + \frac{(M+m)^2v^2}{m\cos^2\beta}\sin^2\beta$$

$$= \frac{M(M+m)}{m}v^2 + \frac{(M+m)^2v^2\sin^2\beta}{m\cos^2\beta}$$

or $$2m^2gl(\cos\beta - \cos\alpha)\cos^2\beta = M(M+m)v^2\cos^2\beta + (M+m)^2v^2\sin^2\beta$$

$$= (M+m)v^2[M\cos^2\beta + (M+m)\sin^2\beta]$$

$$v^2 = \frac{2m^2gl}{M+m} \cdot \frac{(\cos\beta - \cos\alpha)\cos^2\beta}{M + m\sin^2\beta}$$

or $$v = \sqrt{\frac{2m^2gl}{M+m} \cdot \frac{(\cos\beta - \cos\alpha)\cos^4\beta}{M + m\sin^2\beta}}. \quad \textit{Ans.}$$

In the particular case when $\beta = 0$,

$$v = \sqrt{\frac{2m^2gl}{M+m} \cdot \frac{(1-\cos\alpha)}{M}} = \sqrt{\frac{2m^2gl}{M+m} \cdot \frac{2\sin^2\alpha/2}{M}}$$

or $$v = 2m\sin\frac{\alpha}{2}\sqrt{\frac{gl}{(M+m)M}}. \quad \textit{Ans.}$$

31. Let v be the required velocity. It is obtained easily by 'energy conservation'.

Loss in potential energy = Gain in kinetic energy.

$\therefore$ $$\left(\frac{M}{2l} \times x\right)g \times \frac{x}{2} = \tfrac{1}{2}M.v^2$$

or $$v=\sqrt{\frac{gx^2}{2l}}=\sqrt{\frac{g}{2l}}.x. \quad Ans.$$

By Newton's law,

$$\left(\frac{M}{2l}\times x\right)g=\text{Driving force}=Ma$$

or $$a=\frac{g}{2l}x. \quad Ans.$$

Let T be the tension of the chain at the edge.

Then N (reaction at the edge) $=\sqrt{T^2+T^2}=T\sqrt{2}$.

Consider 'free-body' diagram of the part lying on the table. The force acting on it to the right is T and its mass is $\frac{M}{2l}(2l-x)$.

$$\therefore \quad T=\frac{M}{2l}(2l-x)\times a \quad \text{(from force = mass × acc.)}$$

$$=\frac{M}{2l}(2l-x)\times\frac{gx}{2l}=\frac{Mgx(2l-x)}{4l^2}=\frac{Mg}{4l^2}x(2l-x)$$

$$\therefore \quad N=\sqrt{2}T=\frac{Mg}{2\sqrt{2}l^2}x(2l-x). \quad Ans.$$

32. Let v be the velocity of the wedge to the right when the plate goes down by y. The constraint of the system needs that if x be the distance by which the wedge moves to the right, then y and x be related by

$$\frac{y}{x}=\tan\alpha$$

or $$y=x\tan\alpha$$

or $$\dot{y}=\dot{x}\tan\alpha$$

or $$v_y=v\cot\alpha.$$

From conservation of energy,

$$\tfrac{1}{2}Mv^2+\tfrac{1}{2}mv_y{}^2=mgy$$

or $$\tfrac{1}{2}Mv^2+\tfrac{1}{2}m.v^2\tan^2\alpha=mgy.$$

When $y=h$, $$Mv^2+mv^2\tan^2\alpha=2mgh$$

$$v=\sqrt{\frac{2mgh}{M+m\tan^2\alpha}}. \quad Ans.$$

33. Let v be the velocity of m before collision.

By conservation of energy,

$$\tfrac{1}{2}mv_0{}^2+mg(l-l\cos\alpha)=\tfrac{1}{2}mv^2$$

or $$v^2=v_0{}^2+2gl(1-\cos\alpha)=v_0{}^2+4gl\sin^2\frac{\alpha}{2}.$$

(a) By conservation of momentum, $mv = mv' + 2mv''$

or $$v - v' = 2v''.$$

By conservation of kinetic energy,

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}mv'^2 + \tfrac{1}{2}.2mv''^2 \text{ or } v^2 - v'^2 = 2v''^2.$$

Solving, $v'' = \frac{2}{3}v$ and $v' = -\frac{1}{3}v$.

Let m go up by h.

Then $mgh_1 = \frac{1}{2}mv'^2 = \frac{1}{2}m.\frac{1}{9}v^2 = \frac{1}{18}m(v_0^2 + 4gl\sin^2\alpha/2)$

$$h_1 = \frac{1}{18g}\left(v_0^2 + 4gl\sin^2\frac{\alpha}{2}\right). \quad Ans.$$

Let $2m$ go up by h_2.

$$2mgh_2 = \tfrac{1}{2}.2mv''^2 = m.\tfrac{4}{9}v^2$$

$$h_2 = \frac{2}{9g}\left[v_0^2 + 4gl\sin^2\frac{\alpha}{2}\right]. \quad Ans.$$

(b) By conservation of momentum,

$$mv = 3mv' \text{ or } v' = \tfrac{1}{3}v.$$

Let the combined mass go up by h_3.

$$3mgh_3 = \tfrac{1}{2}.3mv'^2$$

or $$gh_3 = \tfrac{1}{2}.\tfrac{1}{9}v^2 = \tfrac{1}{18}v^2$$

or $$h_3 = \tfrac{1}{18}g\left[v_0^2 + 4gl\sin^2\frac{\alpha}{2}\right]. \quad Ans.$$

34. Let the cord extend by ϵ. Then by conservation of energy,

$$mg(l+\epsilon) = \tfrac{1}{2}k\epsilon^2$$

$$k\epsilon - 2mg\epsilon + 2mgl = 0$$

$$\epsilon = \frac{2mg \pm \sqrt{4m^2g^2 + 4k.2mgl}}{2k}$$

or $$\epsilon = \frac{mg}{k} \pm \frac{mg}{k}\sqrt{1 + \frac{2kl}{mg}}.$$

Discarding the meaningles negative sign,

$$\epsilon = \frac{mg}{k}\left(1 + \sqrt{1 + \frac{2kl}{mg}}\right). \quad Ans.$$

Solutions to Test Questions

1. (i). $m_1v = m_1v_1 + m_2v_2$

or $m_1(v - v_1) = m_2v_2$, or $v - v_1 = \dfrac{m_2v_2}{m_1}$

$$\tfrac{1}{2}m_1v^2 = \tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2, \text{ or } m_1(v - v_1)(v + v_1) = m_2v_2^2$$

$$\therefore\ v + v_1 = v_2.$$

Hence $$v_2=\frac{2m_1v}{m_1+m_2};\quad v_1=\frac{m_1-m_2}{m_1+m_2}v.$$

The required fraction$=\dfrac{\frac{1}{2}m_2v_2^2}{\frac{1}{2}m_1v^2}=\dfrac{m_2v_2^2}{m_1v^2}=\dfrac{4m_1m_2}{(m_1+m_2)^2}.$

So the answer is (c).

(ii). (c). (iii) (c). $P=Fv\cos\theta=360\times\dfrac{10\times1000}{3600}\times\cos60^\circ$

$$=500\text{ W.}$$

(iv) (d). $$K_{man}=\tfrac{1}{2}K_{boy}.$$

or $$\tfrac{1}{2}Mv^2=\tfrac{1}{2}\left(\tfrac{1}{2}\frac{M}{2}u^2\right)$$

or $$v^2=\frac{u^2}{4}.$$

Again $$K_{man}=K_{boy}.$$

$$\tfrac{1}{2}M(v+1)^2=\tfrac{1}{2}\frac{M}{2}u^2$$

$$(v+1)^2=\frac{u^2}{2}$$

$$\therefore\quad (v+1)^2=2v^2,\quad\text{or } v=(\sqrt{2}+1)\text{ m/s.}$$

(v). (c). $$F-mg=m\times0\quad\text{or}\quad F=mg$$

$$P=Fv\cos\theta=mgv\cos0^\circ$$
$$=30\times10\times{\cdot}6=180\text{ watt.}$$

(vi). (d). relative velocity before (to the right) $=v-u$

relative velociiy after collision (to the right) $=v'-u.$

By Newton's law of collision,

$$v'-u=-(v-u)=-v+u$$

or $$v'=2u-v=-(v-2u)\text{ towards the wall}$$
$$=v-2u\text{ away from the wall.}$$

(vii). (b).

(viii). (c). Let F be the force in general.

$F-mg=ma$ where $a=$ upward acceleration produced by it.

$$v^2=2.a.h.$$

Work done by gravity $=mg(-h)$.

Work done by the force $=W$ (say).

By 'work-energy' theorem,

$$W+(-mgh)=\tfrac{1}{2}mv^2=\tfrac{1}{2}m.2ah$$

or $$W=mgh+mah=(mg+ma)h=F.h.$$

Thus work done does not depend on the mass on which the force acts. So it does the same work on both the bodies.

(ix). (a)+(b).

(x). (d).

2. By 'momentum conservation',

$$m_1u=m_1v_1+m_2v_2$$

or $$m_1(u-v_1)=m_2v_2.$$

By 'kinetic energy conservation',

$$\tfrac{1}{2}m_1u^2=\tfrac{1}{2}m_1v_1^2+\tfrac{1}{2}m_2v_2^2$$

or $$m_1(u-v_1)(u+v_1)=m_2v_2^2.$$

Combining, $$u-v_1=\frac{m_2v_2}{m_1}$$

$$u+v_1=v_2.$$

Hence, $$v_2=\frac{2m_1u}{m_1+m_2} \text{ and } v_1=\frac{m_1-m_2}{m_1+m_2}\times u.$$

Fraction of energy transferred to B from A

$$=\frac{\tfrac{1}{2}m_2v_2^2}{\tfrac{1}{2}m_1u^2}=\frac{4m_1m_2}{(m_1+m_2)^2}.$$

If $k=\dfrac{m_1}{m_2}$, then the fraction $=\dfrac{4km_2^2}{m_2^2(1+k)^2}=\dfrac{4k}{(1+k)^2}.$

If $k=\dfrac{m_2}{m_1}$, then the fraction $=\dfrac{4km_1^2}{m_1^2(1+k^2)}=\dfrac{4k}{(1+k)^2}.$ *Proved.*

3. Referring all the velocities to the right we have by 'conservation of momentum to the right',

$$10\times9+20\times6=10\times6+20v_2$$

or $$v_2=7{\cdot}5 \text{ m/s}.$$

Relative velocity of A before collision $=9-6=3$ m/s.

Relative velocity of A after collision $=6-7{\cdot}5=-1{\cdot}75$.

By Newton's law of collision,

$$-1{\cdot}75=-e\times3$$

or $$e=\frac{1{\cdot}75}{3}={\cdot}58. \text{ Ans.}$$

Loss in kinetic energy

$=(\tfrac{1}{2}\times10\times9^2+\tfrac{1}{2}\times20\times6^2)-(\tfrac{1}{2}\times10\times6^2+\tfrac{1}{2}\times20\times7{\cdot}5^2)$

$=(405+360)-(180+562{\cdot}5)$

$=22{\cdot}5$ joule. *Ans*

This energy is converted into heat.

4. (a) Velocity on reaching the ground $=v$ (say).

$$v^2-v_0^2=2gh \quad\text{or}\quad v^2=v_0^2+2gh.$$

Let v' be the velocity after collision. Then

$$v'^2=2gh \quad \text{(by } v^2-u^2=2as \text{ formula)}$$

and $$\frac{\frac{1}{2}mv'^2}{\frac{1}{2}mv^2}=\frac{1}{2} \text{ (given)}$$

or $$v'^2=\tfrac{1}{2}v^2$$

$$\therefore \quad 2gh=\tfrac{1}{2}v^2 \text{ or } v^2=4gh$$

$$\therefore \quad 4gh=v_0^2+2gh$$

$$v_0=\sqrt{2gh}=\sqrt{2\times 9{\cdot}8\times 10}=14 \text{ m/s.} \quad \textit{Ans.}$$

(b) When it is thrown upward, initial velocity is $-v_0$.

$$\therefore \quad v^2-(-v_0^2)=2gh, \text{ or } v^2-v_0^2=2gh.$$

It is the same formula. So it will rise to the same height, namely, 10 m.

5. By 'momentum conservation',

$$\cdot 05\times v_0=(5+\cdot 05)v \text{ or } v=\frac{\cdot 05\,v_0}{5{\cdot}05}=\frac{v_0}{101}.$$

Kinetic energy of the interlocked body
= work done against frictional force.

$$\therefore \quad \tfrac{1}{2}\times 5{\cdot}05\times\left(\frac{v_0}{101}\right)^2=0{\cdot}2\times 5{\cdot}05g\times 5$$

$$v_0=101\sqrt{2\times 9{\cdot}8}=447 \text{ m/s.} \quad \textit{Ans.}$$

6. Let v be the velocity at the demarcation (A) of change of roughness.

Then by W-E theorem,

$$\tfrac{1}{2}mv^2-\tfrac{1}{2}mV^2=mg(-h)-\mu_1 mg\cos\alpha.h\operatorname{cosec}\alpha$$

or $$v^2=V^2-2gh-2(2\tan\alpha)gh\cos\alpha\operatorname{cosec}\alpha$$
$$=V^2-2gh-4gh=V^2-6gh.$$

Let H be the vertical height above the demarcation line. Then

$$\tfrac{1}{2}mv^2=mgH+\mu_2 mg\cos\alpha H\operatorname{cosec}\alpha$$
$$=mgH+\tfrac{1}{2}\tan\alpha.mgH\cos\alpha.\operatorname{cosec}\alpha$$

or $$v^2=3gH.$$

$$\therefore \quad 3gH=V^2-6gh. \qquad \text{.. (i).}$$

Let v' be the velocity at the demarcation line after return to A.

Then $$\tfrac{1}{2}mv'^2=mgH-\mu_2 mg\cos\alpha H\operatorname{cosec}\alpha$$
$$=mgH-\tfrac{1}{2}\tan\alpha.mgH\cos\alpha.\operatorname{cosec}\alpha$$

or $$v'^2=2gH-gH=gH$$

or $$3v'^2=3gH$$

$$\therefore \quad 3v'^2=V^2-6gh \qquad \text{.. (ii).}$$

Let the particle go down by vertical distance h'.

Then $$-\tfrac{1}{2}mv'^2 = mgh' - \mu_1 mg\cos\alpha h'\operatorname{cosec}\alpha$$
$$= mgh' - 2\tan\alpha mgh'\operatorname{cosec}\alpha$$
or $$-v'^2 = 2gh' - 4gh' = -2gh'$$
or $$v'^2 = 2gh'$$
or $$3v'^2 = 6gh'$$
or $$6gh' = V^2 - 6gh.$$

It will never return to *O* if $h' \leqslant h$

or $\dfrac{V^2 - 6gh}{6g} \leqslant h$, or $\dfrac{V^2}{6g} - h \leqslant h$, or $V^2 \leqslant 12gh$. *Proved.*

7. By 'conservation of momentum',
$$mv_0 = (m+M)V.$$

By 'conservation of mechanical energy',
$$\tfrac{1}{2}(m+M)V^2 = (m+M)gd.$$
$$\therefore \quad mv_0 = m+M)\sqrt{2gd}$$

or $$v_0 = \frac{m+M}{m} \times \sqrt{2gd}. \quad \textit{Ans.}$$

LESSON 5

Topics : **Circular Motion**
Simple Harmonic Motion

5.1. Circular Motion

Dynamics of Circular Motion. If a particle moves in a circle with uniform speed v and ω is its uniform angular speed, then $v=\omega.r$ where r is the radius of the circle.

Unit of ω is radian/sec.

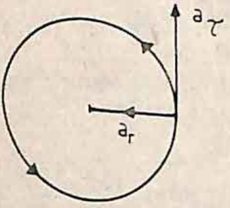


Fig. 5.1

The circular motion is called uniform when the speed v of the particle is constant, and non-uniform when v is variable.

When a particle moves in a circle with variable speed, it has both tangential acceleration and normal or centripetal acceleration. These are given by

Tangential Acceleration : $a_\tau=\frac{dv}{dt}$

where v is its instantaneous speed.

Centripetal Acceleration : $a_r=\frac{v^3}{r}$. This is directed towards the centre.

Centripetal Force = mass × centripetal acceleration $=\frac{mv^2}{r}$.

Tangential Force = mass × tangential acceleration $=ma\tau=m\frac{dv}{dt}$.

Note. To solve problems on circular motion, always remember that the particle in circular motion is an accelerated system and there must be unbalanced force along the normal towards the centre.

This is true both for uniform and non-uniform circular motions. In addition, there will be an unbalanced tangential force if the particle is not in uniform circular motion.

Follow the following steps :

(a) Consider first all the real forces (forces accountable to the environments) on the particle that is, make a 'free-body' diagram of it.

(b) Then resolve all the forces along the normal towards the centre and along the tangent. Take their algebraic sum and for each direction use 'force = mass × acceleration'.

(c) Apply the principle of conservation of energy.

Solve the equations set up for the sought quantity in the problem.

Inclination of cyclists with the vertical or banked roads with the horizontal (called the angle of banking),

$$\tan\theta = \frac{v^2}{rg}.$$

D'Alembert's Principle. According to this principle a dynamic problem can be reduced to a problem of a static-body by introducing a force called 'inertia force' equal to the mass and acceleration at its centre of mass in the direction opposite to the direction of motion in addition to the real forces (forces accountable to its environments). In context of circular motion its importance is that to find a physical quantity we can extend this formalism and assume the body to be at rest. Though it can be used in all prolems on circular motion, we will use it only in specific problems where its use is particularly advantageous.

5.2 Simple Harmonic Motion

A particle (P) undergoes linear SHM when it moves along a straight line (AB) such that its acceleration is always proportional to its displacement (x) from a fixed point (O) on the line, and is directed towards that point. The point is called the 'mean position'. The distance $OA = OB = a$ is called the 'amplitude'.

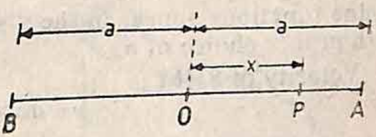


Fig. 5.2

Note. Do not confuse between 'direction of motion' and 'direction of acceleration'. When P lies between O and A, and is moving towards A, it is moving to the right but its acceleration is towards the left.

Since the acceleration (f) fs proportional to the displacement (x), the equation to a SHM is

$$f = -\mu x = -\omega^2 x.$$

Here $\mu = \omega^2 =$ constant. The quantity ω is called the 'cyclic frequency' of SHM, having unit rad/sec, and is obtained from

$$\omega = \sqrt{\frac{\text{acceleration}}{\text{displacement}}}.$$

It is related to the time period of SHM as

$$T=\frac{2\pi}{\omega}.$$

Note. To prove that a particular motion is simple harmonic, merely establish that $f \propto x$, and is directed towards the mean position.

Displacement of SHM :

$$x=a\sin(\omega t+\alpha)$$

where a is the 'amplitude' and α is another constant called the 'epoch' of SHM.

The choice of α depends on the starting point of the SHM, e.g.

Starting point	Value of α	Equation for displacement
A	$\frac{\pi}{2}$	$x=a\sin\left(\omega t+\frac{\pi}{2}\right)=a\cos\omega t$
O	0	$x=a\sin\omega t$
B	$-\frac{\pi}{2}$	$x=a\sin\left(\omega t-\frac{\pi}{2}\right)=-a\cos\omega t$

For starting points other than these, values of α intermediate between 0 and $\frac{\pi}{2}$ are obtained. It will be noticed that both sine and cosine functions appear in the displacement. Either may be used, with proper choice of α.

Velocity of SHM :

$$v=a\omega\cos(\omega t+\alpha)$$
$$=\omega\sqrt{a^2-x^2}.$$

When 'epoch' is zero, $v=a\omega\cos\omega t$.

Acceleration of SHM :

$$f=-a\omega^2\sin(\omega t+\alpha)=-\omega^2\times x.$$

When 'epoch' is zero, $f=-a\omega^2\sin\omega t$.

Energy of SHM :

U (potential energy) $=\frac{1}{2}m\omega^2x^2$.
T (kinetic energy) $=\frac{1}{2}m\omega^2(a^2-x^2)$.
E (total energy) $=\frac{1}{2}ma^2\omega^2$.

Phase of SHM and Phase angle. The phase of a particle in SHM is 'its state of motion' determined by displacement and velocity. If both displacement and velocity are identical, then we say that phases are identical. The angle associated with sine or cosine in the displacement and velocity expressions is the same and it is $(\omega t+\alpha)$. This angle is called 'the phase angle'.

Generating Circle of SHM. A simple harmonic motion may be considered as the notion of the projection (P) of a particle (Q) moving in a circle of radius (a) equal to the amplitude of the given SHM, with angular speed equal to the cyclic frequency ω of the given SHM. This circle is called 'generating circle'. In some problems we will take the help of generating circle because that will enable to find solution in a simpler way.

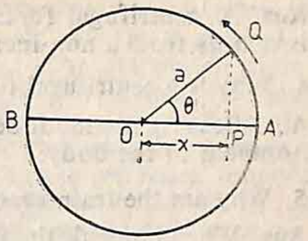


Fig. 5.3

Lissajous Figures. The figure traced by a particle subjected to two simple harmonic motions at right angles having frequencies in a simple ratio. If the figure is tangentially touched by a horizontal line (x-axis) at n_x points and at n_y points by a vertical line, the frequencies of the simple harmonic motions are in the inverse ratio of these numbers. That is,

$$\frac{f_x}{f_y}=\frac{n_y}{n_x}.$$

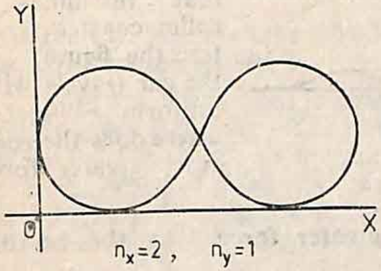


Fig. 5.4

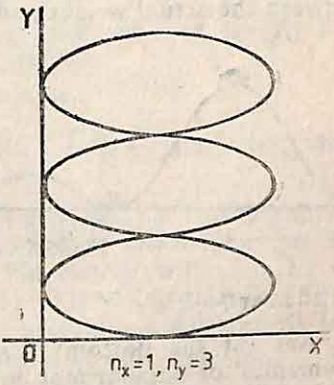


Fig. 5.5

CRITICAL QUESTIONS

1. Is a body in circular motion in equilibrium ?

Ans. No. A body is said to be in equilibrium when its acceleration is zero. A body in circular motion is an accelerated body and so it is not in equilibrium.

2. Is centripetal force a special kind of force ?

Ans. No. It is simply the name of a force which provides centripetal acceleration. For example, a tension in the string by which a stone is whirled is the centripetal force.

3. Is centrifugal force a real force or a pseudo force ?

Ans. A centrifugal force is a pseudo force. Its existence is realised only from a non-inertial frame rotating with the body.

4. Why is a centrifugal force called a pseudo force ?

Ans. Because it cannot be attributed to any of the bodies in the environment of the body.

5. Why are the train road beds and highways banked on curves ?

Ans. When banked, the normal reactions on the wheels will give rise to component forces towards the centre of the path which is essential for maintenance of motion of the train or car in circular path. In the absence of banking the necessary centripetal force is provided by lateral pressure on the outer wheels. This involves wear and tear of tyre or wheel in addition to chances of accident at high speeds.

6. How does the earth's rotation affect the apparent weight of a body at the equator ?

Ans. The earth's rotation reduces the weight of a man. A part of the weight provides the necessary centripetal force. At any point on the surface of the earth the apparent weight is the vector difference between the actual weight and the centripetal force.

7. A car is riding on a country road that resembles a roller coaster track (see the figure). If the car travels with uniform speed, where does the road exert greater force on the car ?

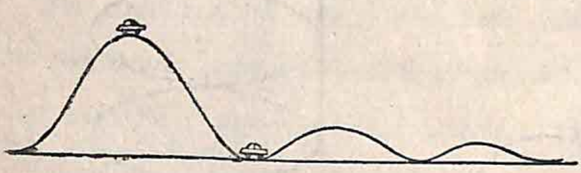


Fig. 5.6

Ans. At the bottom it exerts greater force. At the bottom 'dynamics' of circular mot.on needs

$$N - mg = \frac{mv^2}{r}, \quad \text{or} \quad N = \frac{mv^2}{r} + mg$$

and at the top $mg - N' = \frac{mv^2}{r}$, or $N' = mg - \frac{mv^2}{r}$;

$$\therefore \quad N > N'.$$

8. A passenger in a car finds himself sliding towards the left door as the driver makes a sudden turn to the right. Describe the forces on the passanger at that instant if (a) the motion is viewed from a reference frame attached to the earth and (b) if attached to the car.

Ans. In the reference frame attached to the earth forces acting on the passenger are (i) *mg*, weight of the passenger vertically downward, (ii) *N*, normal reaction of the seat upward, (iii) the frictional force towards the centre of the curved path. The frictional force can increase only up to its limiting value which is μN. At the sharp turn (radius small) large centripetal force is needed to maintain the motion along the curved path. If the force needed be greater than the *limiting frictional force*, an observer in the reference frame finds the passenger to move *tangentially* on to greater circles till he reaches the door. The door now exerts necessary centripetal force on the passenger. In the reference frame attached to the car, the forces acting on the passenger are : (i) *mg*, vertically downward, (ii) *N*, reaction of the seat upwards, (iii) the frictional force towards the centre and (iv) the 'inertia force' i.e. 'the centrifugal force' away from the centre.

Under the action of these forces the passenger is at rest relative to an observer in this frame of reference, say, another passenger in the car. When the inertia force i.e. the centrifugal force exceeds the *limiting frictional force*, the second passenger will find him to move *normally away* to the outer door till he reaches the door. The door now exerts force on the passenger and balances the 'inertia force' and thus again he appears stationary to the second passenger.

9. A spirit level is tied to a string and whirled rapidly in a horizontal plane. Where will the bubble lie ?

Ans. The bubble will lie at the inner edge.

10. Can you examine whether a table top in a train is truly horizontal by a spirit level ?

Ans. No. If the road is straight but inclined, the bubble shifts to one extreme position. The bubble will shift to one extreme position if the road is horizontal but curved. The bubble will shift to one extreme position if the road is straight and level but the table top is not horizontal. Hence it cannot be ascertained by a spirit level whether a table top in a train is horizontal or not.

11. A coin is put on the turn table of record player. The motor is started, but before final speed of rotation is reached, the coin flies off. Explain.

Ans. In the beginning the frictional force provides the necessary centripetal force on the coin. With the increase of speed, the frictional force also increases. This continues up to limiting friction. The moment limiting friction is reached, and exceeded, the coin flies off tangentially relative to an observer on the ground, the limiting frictional force being unable now to provide the necessary centripetal force.

12. A particle moving uniformly in a circle is in rotational equilibrium. Is it true or false ?

Ans. It is true. A particle is said to be in rotational equilibrium when its angular acceleration is zero. Since the particle moves uniformly, its angular velocity is constant in magnitude and direction and so its angular acceleration is zero. So it is true that a particle in a uniform circular motion is in rotational equilibrium.

13. What is the power needed to maintain a uniform circular motion ?

Ans. No power is needed to maintain uniform circulat motion.

$$\text{Power needed} = \vec{F} \cdot \vec{v} = Fv\cos\frac{\pi}{2} = 0.$$

14. Is any work done by centripetal force ?

Ans. No. Work done by a force $= \vec{F} \cdot \vec{x}$.

$$\text{Work done by centripetal force} = Fx\cos\frac{\pi}{2} = 0.$$

15. What is the torque due to centripetal force about the axis of rotation ?

Ans. Torque $(\tau) = F.d = F.0 = 0$.

16. How does the pressure in a rotating liquid change from the axis to the wall of the vessel ?

Ans. Unlike the liquid at rest, the pressure of a rotating liquid *increases* from the axis to the wall. In a liquid at rest pressure is the same at all points on the same horizontal line. For rotational motion an inwardly directed force is a must. In a rotating liquid this force is provided by the pressure difference. Without 'horizontal pressure gradation' from the axis to the wall, the 'rotational equilibrium' of the liquid mass is not possible. Consider a cylindrical layer of liquid of inner radius x and outer radius $x+dx$. Let p be the pressure of the rotating liquid at a distance x from the axis and $p+dp$ at a distance $x+dx$.

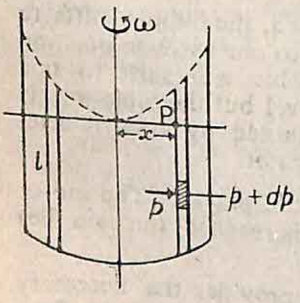


Fig. 5.7

The inward force on the layer of liquid

$= 2\pi$,

height of the liquid

$= 2\pi x l dp$ where l is the

or $= (2\pi x l dx \rho) \times \omega^2 x$

$$dp = \rho\omega^2 x \, dx.$$

Integrating, $p = \frac{1}{2}\rho\omega^2 x^2 + p_0$ where p_0 is the pressure at the axis.

17. Why the shape of a rotating liquid surface paraboloidal in shape ?

Ans. In a rotating liquid there is a pressure gradation from the axis to the wall. The pressure at a distance x from the axis is given by

$$p = p_0 + \tfrac{1}{2}\rho\omega^2 x^2.$$

Consider a point P at distance x from the axis and at a depth y from the liquid surface. Then

$$p = p_0 + \rho g y$$

$$\therefore \quad p_0 + \rho g y = p_0 + \tfrac{1}{2}\rho\omega^2 x^2$$

or

$$y = \tfrac{1}{2}\frac{\omega^2}{g}x^2, \quad \text{or} \quad x^2 = \frac{2g}{\omega^2} \times y.$$

Comparing this equation with the standard form of equation of a parabola $y^2 = 4ax$, we find that the rotating liquid surface is a parabola about the vertical axis of rotation and the entire surface is, therefore, paraboloidal.

18. Why is cream separated from skimmed milk ?

Ans. Let us consider an elementary body B suspended in the liquid. Let it be of face area $\triangle s$ at a distance x from the axis and let its thickness be dx. Then the inward force on it due to 'horizontal pressure gradation' $= \triangle s.dp$. But $dp = \rho\omega^2 x\, dx$ (see deduction in Q. 16).

$$\therefore \quad \triangle F_{inward} = \triangle s \rho\omega^2 x\, dx.$$

The 'inertia force', i.e., centrifugal force needed for rotational equilibrium of the particle relative to the rotating liquid $= (\triangle s.dx\sigma)\omega^2 x$ where σ is the density of the suspended particle.

$$\therefore \quad \frac{\triangle F_{inward}}{\triangle F_{centrifugal}} = \frac{\rho}{\sigma}.$$

$\therefore$ If $\rho > \sigma$, $\triangle F_{inward} > \triangle F_{centrifugal}$

and if $\rho < \sigma$, $\triangle F_{inward} < \triangle F_{centrifugal}$.

Thus, particles whose density is less than that of the liquid are driven towards the axis of rotation and those whose density is greater than that of the liquid are driven away from the axis. Cream is lighter than milk and so it is separated from the milk and collects at the axis.

19. Can force determine the direction of motion ? direction of acceleration ?

Ans. No, a force cannot determine the direction of motion. Yes, a force simply determine the direction of acceleration. For example, in circular motion force is directed to the centre but the motion is along the tangent, but acceleration is definitely along the normal which is the direction of the force.

20. From a big rotating platform a loose bolt at the circumference gets detached. In what direction will an observer on the platform see it to move out ? In what direction will an observer on the fixed ground see it to move out ?

Ans. The first observer (*A*) will see it to move out normally away from the centre of the platform, while the second observer (*B*) sees it to move out tangentially in the direction of motion of the platform.

21. Is velocity of a particle moving uniformly in a circle constant or variable ? Justify.

Ans. It is variable. Fix a frame of reference at the centre of the circle. The speed is constant but direction of motion changes with respect to x-axis. So the velocity is time-dependent.

22. Is acceleration of a particle in a uniform circular motion constant or varible ? Explain.

Ans. Here also magnitude is always the same, viz., v^2/r but the direction with respect to x-axis changes from instant to instant and so the acceleration is also time dependent.

23. A motor cyclist makes a show of going in vertical circle without falling inside an iron cage. Explain his rotational equilibrium from the point of view of an observer fixed to the ground and that of an observer in the frame fixed to the motor cycle.

Ans. From the point of view of an observer fixed on the ground, the rotational equilibrium is due to vertically downward reaction of the iron cage on the tyre of the motor cycle at the highest point. The centripetal force necessary for rotational equilibrium is provided by the reaction R and the weight of the (motorcycle+rider). From the point of view of an observer fixed to the motor cycle, the system is in equilibrium due to balancing of the reaction of the iron cage plus the weight by the 'inertia force', i.e., the centrifugal force.

24. When a car passes at high speed over a curved bridge across a river, will it leave the ground at the higest point against the gravity (vertically) or leave it tangentially at that point. Explain.

Ans. To an observer fixed to the ground the car will leave the ground tangentially as the normal reaction is insufficient to provide necessary centripetal force and so due to inertia the car leaves the path tangentially. To an observer fixed to the car, say, a passenger, the car will appear to leave the ground and move away from the centre.

25. In an unbanked curved road a car will skid if it is slippery and will tend to overturn if road is rough and speed is high. Explain.

Ans. In an unbanked curved road, frictional force provides the necessary centripetal force. If road is slippery, it cannot provide much centripetal force and so the car is likely to skid. On the other hand if the road is rough, there will arise enough frictional force to provide centripetal force. But its moment about the cg of car may exceed the moment of the couple formed by its weight and the reaction on the outer wheel. So if the road is rough the car is likely to overturn at high speeds.

26. In a magnetic field a charged particle always moves in a circular path. Why ?

Ans. In a magnetic field a charged particle always experiences forces at right angles to the direction of motion which is the characteristic of a circular motion. So a charged particle always moves in circular path in a magnetic field.

OBJECTIVE QUESTIONS

1. When a ball is whirled in a circle and the string supporting the ball is released, the ball flies off tangentially. This is due to (a) action of centrifugal force, (b) due to inertia for linear motion, (c) due to centripetal force, (d) due to some unknown cause.

Ans. (b).

2. A particle of mass $\frac{1}{2}$ kg is whirled in a horizontal circle of radius 1 m with a constant speed of 4 ms^{-1}. The tension of the string is (a) 1 N, (b) 2 N, (c) 3 N, (d) 4 N.

Ans. (d).

3. The breaking force of a string is 16 kgf. The maximum frequency at which a body of mass $\frac{1}{2}$ kg can be whirled in a horizontal smooth plane ($g=\pi^2$ ms^{-2}) with a length $\frac{1}{2}$ m of that string is (a) 10, (b) 5, (c) 4, (d) 8.

Ans. (c).

4. A spirit level is placed with its centre at the axis of a turntable. The bubble will lie (a) at the centre, (b) at the outer edge, (c) at the inner edge, (d) will oscillate about the centre.

Ans. (a).

5. A spirit level is placed at the edge of a turn table along its radius. The bubble will lie (a) at the centre of the container, (b) at the outer edge of the container, (c) at the inner edge of the container, (d) will oscillate about the centre of the container.

Ans. (c).

6. A string of length l is used to whirl a particle in a vertical plane. The ratio of the maximum speed to the minimum speed in the circle when it just goes through complete revolution is

(a) 4 : 1, (b) $\sqrt{3}$: 1, (c) $\sqrt{5}$: 2, (d) $\sqrt{5}$: 1.

Ans. (d).

7. Which one of the following represents a simple harmonic

motion ? (a) $x^2=a+bv$, (b) $x=\sqrt{a+bv^2}$, (c) $x=a-bv$, (d) $x=\sqrt{a-bv^2}$ where a and b are two positive constants.

Ans. (d).

8. In a simple harmonic motion the displacement and velocity differ in phase by (a) 0, (b) $\pi/2$, (c) π, (d) $\pi/4$.

Ans. (b).

9. A particle in simple harmonic motion has the same displacemets equal to half the amplitude but opposite velocities at two instants. The minimum time interval between the two instants is
(a) $T/2$, (b) $T/3$, (c) $T/4$, (d) $T/6$.

Ans. (d).

10. A disc revolves in a horizontal plane at a steady rate of 3rad/s. A coin will remain on the disc if kept at a distance of 20 cm from the axis of rotation. The coefficient of friction ($g=\pi^2$ ms^{-2}) is
(a) ·5, (b) ·3, (c) ·20, (d) ·72.

Ans. (d). The centripetal force $=\mu mg=m\omega^2 r$

or $$\mu=\frac{\omega^2 r}{g}=\frac{4\pi^2 n^2 r}{\pi^2}=4\times 9\times\cdot 02=\cdot 72.$$

11. The correct angle of banking for a curved road of radius 12 cm for a speed of 108 kph ($g=10$ ms^{-2}) is
(a) 45°, (b) 30°, (c) 37°, (d) 60°.

Ans. (c). $$\tan\theta=\frac{v^2}{r.g}.$$

$$v=108 \text{ kph}=30 \text{ ms}^{-1}.$$

$$\therefore\ \tan\theta=\frac{900}{120\times 10}=\frac{3}{4}=\cdot 75.$$

$$\therefore\ \theta=37°.$$

12. When a particle is moved in vertical circle, (a) it has constant radial and tangential acceleration, (b) it has variable tangential and radial acceleration, (c) it has only constant radial acceleration, (d) it has only constant tangential acceleration.

Ans. (b). Reason : When the particle is moved in a vertical circle the force towards the centre and along the tangent vary from point to point and so its radial and tangential accelerations vary.

13. A body of mass 4 kg is moved in verical circle with sufficient speed. Its tangential acceleration ($g=10$ ms^{-2}) when the string makes an angle 30° with the downward vertical is
(a) 4 ms^{-2}, (b) 5 ms^{-2}, (c) 7·5 ms^{-2}, (d) 8 ms^{-2}.

Ans. (b). The tangential force $=mg\sin\theta=ma_\tau$

$$\therefore\ a_\tau=g\sin\theta=10\times\sin 30°=5 \text{ ms}^{-2}.$$

14. If the rotation of the earth were such that $g=0$ at the equator, the length of the day would be ($g=10\ \text{ms}^{-2}$ at the equator and take $\pi=3$ and radius of the earth $=6400$ km)

(a) 1 hour, (b) $\frac{3}{4}$ hour, (c) $\frac{4}{3}$ hour, (d) $\frac{5}{4}$ hour.

Ans. (c). The apparent weight of a body on the surface of the earth is the vector difference between gravitational pull and the centripetal force. If the speed of rotation were so great that the centripetal force equalled the gravitational pull, then the apparent weight of a body would be zero, that is, g would be apparently zero.

$$\therefore\quad mg=m\omega^2R;\quad \omega=\sqrt{g/R}$$

or $$T=2\pi\sqrt{\frac{R}{g}}=2\times3\sqrt{\frac{6400\times1000}{10}}=4800\ \text{sec}=\tfrac{4}{3}\ \text{hr. Proved.}$$

15. A point traversed half a circle of radius R during the interval τ. The mean speed during the interval is

(a) $\frac{2\pi R}{\tau}$, (b) $\frac{2R}{\tau}$, (c) $\frac{\pi R}{\tau}$, (d) $\frac{R}{\tau}$.

Ans. Mean speed $=\dfrac{\text{Distance covered}}{\text{time}}=\dfrac{\pi R}{\tau}$.

So the answer is (c).

16. In the above example, the modulus of the mean velocity is

(a) $\frac{2\pi R}{\tau}$, (b) $\frac{2R}{\tau}$, (c) $\frac{\pi R}{\tau}$, (d) $\frac{R}{\tau}$.

Ans. $$<\vec{v}>=\frac{\Delta\vec{r}}{\Delta t}=\frac{\vec{r}_f-\vec{r}_i}{\Delta t}=\frac{R\vec{i}-(-R\vec{i})}{\Delta t}=\frac{2R\vec{i}}{\tau}\quad(\because\ \Delta t=\tau)$$

$\therefore\quad |<\vec{v}>|=\dfrac{2R}{\tau}$, so the answer is (b).

***17.** In the above examples, if the point moved with constant tangential acceleration, the modulus of mean acceleration is

(a) $\frac{2\pi R}{\tau^2}$, (b) $\frac{2R}{\tau^2}$, (c) $\frac{\pi R}{\tau^2}$, (d) $\frac{R}{\tau^2}$.

Ans. $$<\vec{a}>=\frac{\vec{v}_f-\vec{v}_i}{\tau}=\frac{v'(-\vec{j})-v\vec{j}}{\tau}=-\frac{(v'+v)}{\tau}\vec{j}$$

$$\therefore\quad |<\vec{a}>|=\frac{v'+v}{\tau}.$$

Since the tangential acceleration is constant, $\dfrac{dv}{dt}=c$ (a constant).

$$\therefore \quad \frac{v+v'}{2} = \text{average speed during the interval} = \frac{\pi R}{\tau}$$

or
$$v+v' = \frac{2\pi R}{\tau}.$$

$$\therefore \quad |<\vec{a}>| = \frac{2\pi R}{\tau^2}.$$

So the answer is (a).

18. A cord is tied to a pail of water and the pail is swing in a vertical circle of radius 4 m ($g=9 \text{ ms}^{-2}$). The minimum velocity of the pail at the highest point of the circle, if no water is to spill from the pail, is (a) 7 ms^{-1}, (b) 6 ms^{-1}, (c) 10 ms^{-1}, (d) 3 ms^{-1}.

Ans. (b). Let N be the downward reaction of the bottom of the pail on an elementary mass m of water. Then force on the mass towards the centre at the highest point is $N+mg$, where m is the elementary mass. For rotational equilibrium of the element of mass we have $N+mg=m\omega^2 r$.

Now $N \geqslant 0$ because N negative means mass is not in contact with the pail.

$$\therefore \quad m\omega^2 r \geqslant mg$$

or
$$v^2 \geqslant gr \quad (\because \quad v=\omega r)$$

$$v \geqslant \sqrt{gr}$$

$$\therefore \quad v_{min} = \sqrt{gr} = \sqrt{9 \times 4} = 6 \text{ ms}^{-1}.$$

19. A 1200-kg automobile rounds a level curve of radius 200 m, on an unbanked road, with a velocity of 72 kph. The minimum coefficient of friction between tyres and road in order that the automobile shall not skid is ($g=10 \text{ ms}^{-2}$)

(a) 0·3, (b) 0·2, (c) 0·6, (d) 0·5.

Ans. (b). In an unbanked road the centripetal force is provided by frictional force.

$$\therefore \quad f_{friction} = \frac{mv^2}{r}.$$

But $f_{lim\ friction} \geqslant f_{friction}$, or $\mu mg \geqslant \dfrac{mv^2}{r}$

or
$$\mu > \frac{v^2}{g.r}$$

$$\therefore \quad \mu_{min} = \frac{v^2}{g.r} = \frac{20 \times 20}{10 \times 200} = 0{\cdot}2.$$

20. The maximum velocity at which a truck can safely travel without toppling over, on a curve of radius 250 m (the height of the centre of gravity of the truck above the ground is 1·5 m and the distance between wheels is 1·5 m, the track being horizontal) is
(a) 30 ms^{-1}, (b) 35 ms^{-1}, (c) 40 ms^{-1}, (d) 45 ms^{-1}.

Ans. (b). The car will not topple so long as the moment of the lateral frictional force providing centripetal force about cg is less than the moment of the vertical reaction on the outer wheel about the same point. That is,

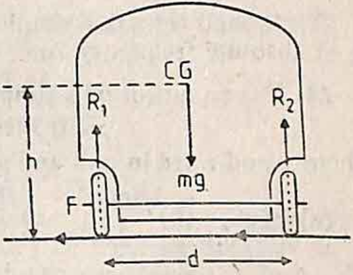


Fig. 5.8

$$F \times h \leqslant R_2 \times \frac{d}{2}$$

$$R_1 + R_2 = mg.$$

At the instant of toppling, the left-hand wheel leaves contact with the ground, so that $R_1 = 0$.

$$\therefore \quad R_2 = mg.$$

But $$P = \frac{mv^2}{r} \text{ and } R_2 = mg.$$

$$\therefore \quad \frac{mv^2}{r} \times h \leqslant mg\frac{d}{2}, \quad \text{or} \quad v \leqslant \sqrt{\tfrac{1}{2}g.\frac{d.r}{h}}.$$

$$\therefore \quad v_{max} = \sqrt{\tfrac{1}{2}\frac{g.d.r}{h}} = \sqrt{\tfrac{1}{2} \times \frac{9{\cdot}8 \times 1{\cdot}5 \times 250}{1{\cdot}5}} = 7 \times 5 = 35 \text{ ms}^{-1}.$$

21. A particle is moving at a speed 40 ms^{-1} on a circular track of radius 400 m. This speed is increasing at the rate of 3 m s^{-2}. The acceleration of the car is
(a) 4 ms^{-2}, (b) 7 ms^{-2}, (c) 5 ms^{-2}, (d) 3 ms^{-2}.

Ans. (c). $a_\tau = 3 \text{ m s}^{-2}$, $a_r = v^2/r = \frac{40^2}{400} = 4 \text{ m s}^{-2}$

$$a = \sqrt{a_\tau^2 + a_r^2} = \sqrt{3^2 + 4^2} = 5 \text{ m s}^{-2}.$$

22. The time period of the simple harmonic motion represented by $f + 4\pi^2 x = 0$, where f and x are instantaneous acceleration and displacement, is (a) $\frac{1}{2}$ s, (b) 4 s, (c) 1 s, (d) 2 s.

Ans. (c). $f = -4\pi^2 x$.

$$\therefore \quad \omega^2 = 4\pi^2, \quad \text{or} \quad \omega = 2\pi, \quad \text{or} \quad \frac{2\pi}{T} = 2\pi, \quad \text{or} \quad T = 1 \text{ s}.$$

23. The potential energy in a simple harmonic motion fluctuates at the frequency of (a) n, (b) $2n$, (c) $3n$, (d) 0 where n is the frequency of the simple harmonic motion.

Ans. (c). $U=\frac{1}{2}m\omega^2x^2=\frac{1}{2}ma^2\omega^2\sin^2\omega t$

$$=\tfrac{1}{4}ma^2\omega^2(2\sin^2\omega t)$$

$$=\tfrac{1}{4}ma^2\omega^2(1-\cos 2\omega t)$$

or

$$U=\tfrac{1}{4}ma^2\omega^2-\tfrac{1}{4}ma^2\omega^2\cos 2\omega t$$

$$=\tfrac{1}{4}ma^2\omega^2+\tfrac{1}{4}ma^2\omega^2\sin\left(2\omega t-\frac{\pi}{2}\right).$$

The second term is a simple harmonic term of cyclic frequency 2ω or of absolute frequency $2n$.

24. The equation of a simple harmonic motion is

$$x=0{\cdot}34\cos(3000t+0{\cdot}74)$$

where x and t are in mm and s. The frequency of the motion is

(a) 3000, (b) $\frac{3000}{2\pi}$, (c) $\frac{0{\cdot}74}{2\pi}$, (d) $\frac{3000}{\pi}$.

Ans. (b).

25. A body of mass m is suspended by a spring of force constant k. When the body is depressed a little and then released, it oscillates with frequency

(a) $\frac{1}{2\pi}\sqrt{\frac{m}{k}}$, (b) $2\pi\sqrt{\frac{m}{k}}$, (c) $\frac{1}{2\pi}\sqrt{\frac{k}{m}}$, (d) $2\pi\sqrt{\frac{k}{m}}$.

Ans. (c).

Let x be the increase in length at time t. Then instantaneous force on the body $=-kx$.

$$\therefore \quad -kx=mf$$

or

$$f=-\frac{k}{m}x$$

$$\therefore \quad \omega^2=\frac{k}{m} \quad \text{or} \quad n=\frac{1}{2\pi}\sqrt{\frac{k}{m}}.$$

26. If the above arrangement is taken to the moon, where the accelaration due to gravity is about $\frac{1}{6}$th that on the earth, how will the frequency of oscillation of the spring be affected.

Ans. It will remain the same. The frequency depends only on the spring constant and on the mass. It does not depend on the acceleration due to gravity.

27. A particle describes simple harmonic motion along a line 4 cm long. The velocity of the particle when it passes through the mean position is 3·142 m/s. The period of the SHM is

(a) ·03 s, (b) ·04 s, (c) ·8 s, (d) ·01 s.

Ans. (b).

28. When the displacement is one-half the amplitude, the fraction of the total energy that is potential is

(a) $\frac{1}{2}$, (b) $\frac{3}{4}$, (c) $\frac{1}{4}$, (d) $\frac{1}{8}$.

Ans. (c). $E=\frac{1}{2}ma^2\omega^2$ and $U=\frac{1}{2}m\omega^2x^2$.

$$\therefore \quad \text{Fraction}=\frac{U}{E}=\frac{x^2}{a^2}=\frac{a^2/4}{a^2}=\frac{1}{4}.$$

29. When the displacement is one-fourth of the amplitude, the fraction of the total energy, that is, kinetic is

(a) $\frac{1}{2}$, (b) $\frac{3}{4}$, (c) $\frac{12}{16}$, (d) $\frac{15}{16}$.

Ans. (d). $E=\frac{1}{2}ma^2\omega^2$; $T=\frac{1}{2}m\omega^2(a^2-x^2)$

$$\therefore \quad \frac{T}{E}=\left(1-\frac{x^2}{a^2}\right)=\left(1-\frac{1}{16}\right)=\frac{15}{16}.$$

30. The maximum velocity of a body undergoing SHM is ·2 m/s and its acceleration at ·1 m from the mean position is ·4 m/s^2. The amplitude of the SHM is

(a) ·25 m, (b) ·3 m (c) ·1 m (d) ·05 m.

Ans. (c).

31. A body of mass 10 gm excutes SHM with amplitude 2 cm and time period 2 s (take $\pi^2=10$). The energy of the particle is

(a) 10 J, (b) 10^{-5} J, (c) 2×10^{-5} J, (d) 4×10^{-5} J.

Ans. (c).

32. Grouping of springs is similar to grouping of capacitors. Is it true or false ?

Ans. It is true. When two springs are connected in parallel, the force constant of the equivalent spring is given by

$$k=k_1+k_2+k_3+\ldots.$$

When they are joined in 'series', the equivalent force constant is given by

$$\frac{1}{k}=\frac{1}{k_1}+\frac{1}{k_2}+\frac{1}{k_3}+\ldots\ldots$$

These are also the rules for grouping of capacitors.

33. A body of mass m is hung from a light horizontal rod which, in its turn, is suspended by two springs of stiffness k_1 and k_2 from the ceiling. The period of oscillation of the mass is

(a) $2\pi\sqrt{\frac{k_1+k_2}{m}}$,

(b) $2\pi\sqrt{\frac{k_1k_2}{(k_1+k_2)m}}$,

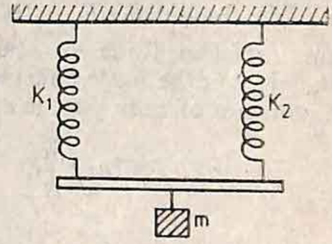


Fig. 5.9

(c) $2\pi\sqrt{\frac{m}{k_1+k_2}}$, (d) $2\pi\sqrt{\frac{(k_1+k_2)m}{k_1k_2}}$.

Ans. (a).

34. A spring of force constant k_1 is clamped at one end to ceiling and its lower end is connected to another spring of force constant k_2. The combination carries a mass m. The period of oscillation is

(a) $2\pi\sqrt{\frac{k_1+k_2}{m}}$, (b) $2\pi\sqrt{\frac{k_1}{k_2m}}$,

(c) $2\pi\sqrt{\frac{k_2}{k_1m}}$, (d) $2\pi\sqrt{\frac{k_1k_2}{(k_1+k_2)m}}$.

Ans. (d).

35. The end of one of the prongs of a tuning fork which executes simple harmonic motion of frequency 1000 Hz has amplitude 0·4 mm. Its maximum acceleration is (take $\pi^2=10$)
(a) 10 ms^{-2}, (b) 3 ms^{-2}, (c) 6·4 ms^{-2}, (d) ·4 ms^{-2}.

Ans. (c).

36. A spring of force constant 19·6 N/m hangs vertically. A body of mass ·4 kg is attached to its free ends. This is depressed slightly and then released. The period of oscillation of the mass is

(a) $\frac{\pi}{3}$ s, (b) $\frac{2\pi}{5}$ s, (c) $\frac{2\pi}{8}$ s, (d) $\frac{2\pi}{7}$ s.

Ans. (d). We have standard formula $T=2\pi\sqrt{\frac{m}{k}}$.

$$\therefore \quad T=2\pi\sqrt{\frac{\cdot 4}{19\cdot 6}}=\frac{2\pi}{7} \text{ s.}$$

37. The period of vibration of a mass suspended by a spring is T. The spring is cut into n equal parts and the body is again suspended by one of the pieces. The time period of oscillation of the mass is
(a) $\sqrt{n}.T$, (b) nT, (c) $T/\sqrt{n}$, (d) n^2T.

Ans. (c). The force constant is inversely proportional to the length. If k is the force constant of the original spring, then the force constant of each part is nk.

$$\therefore \quad T=2n\sqrt{\frac{m}{k}}, \quad \text{and} \quad T'=2\pi\sqrt{\frac{m}{nk}}.$$

$$\therefore \quad T'=\frac{T}{\sqrt{n}}.$$

EXAMPLES

1. A stone of mass 1 kg is attached to one end of a string 1 m long, of breaking strength 500 N, and is whirled in a horizontal circle on a frictionless table top. The other end of the string is kept fixed. Find the maximum speed the stone can attain without breaking the string.

Sol. The forces acting on the stone are : (i) mg, downwards, (ii) N, normal reaction of the table, (iii) T, tension of the string towards the centre.

Resolving all the forces along the horizontal towards the centre of the circle and vertical we have

$$T = m \times \frac{v^2}{r} \quad \text{and} \quad N = mg$$

(there is no acceleration along the vertical).

Obviously, $T_{max} = \frac{mv^2{}_{max}}{l}$ where l is the length of the string.

or $$v^2{}_{max} = \frac{T_{max} \times l}{m} = \frac{500 \times 1}{1} = 500$$

or $$v_m = 22{\cdot}36 \text{ ms}^{-1}. \quad \textit{Ans.}$$

2. (a) How many revolutions per minute must the apparatus of the figure make about a vertical axis in order that the cord shall make an angle of 45° with the vertical ? (b) What is then the tension in the cord ? Given : $l = 1{\cdot}414$ m, $a = 20$ cm, $m = 500$ gm.

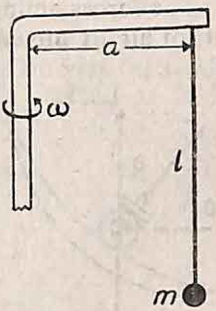


Fig. 5.10

Sol. When the shaft is rotated, the mass m will revolve in a circle centred on the shaft.

The forces acting on the mass : (i) mg, downwards, (ii) T, at inclination $\theta = 45°$ to the vertical.

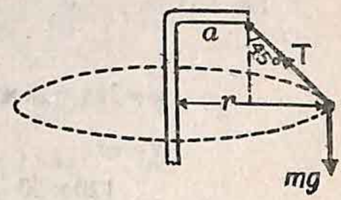


Fig. 5.11

Resolving T along the vertical and horizontal directions, and equating forces

$$T\sin\theta = \frac{mv^2}{r} = \frac{m(r\omega)^2}{r} = m\omega^2 r \quad \text{and} \quad T\cos\theta = mg.$$

Dividing, $\tan\theta = \dfrac{\omega^2 r}{g}$, or $\omega^2 = \dfrac{g\tan\theta}{r}$.

Here $r = a + l\sin\theta = \cdot2 + 1\cdot414 \times \dfrac{1}{\sqrt{2}} = 1\cdot2$ m.

$$\therefore \quad \omega^2 = \frac{9\cdot8 \times \tan 45^\circ}{1\cdot2} = \frac{9\cdot8}{1\cdot2}$$

or $$\omega = \sqrt{\frac{9\cdot8}{1\cdot2}} = 2\cdot86 \text{ rad/sec}$$

or $$n = \frac{\omega}{2\pi} = \frac{2\cdot86}{2\times\pi} = \cdot46/\text{sec}$$

$$= \cdot46 \times 60 = 27\cdot6 \text{ per minute.} \quad \textit{Ans.}$$

(b) $T = \dfrac{mg}{\cos\theta} = \dfrac{\frac{1}{2}\times9\cdot8}{\cos45^\circ} = \dfrac{9\cdot8}{\sqrt{2}} = 6\cdot93$ N. *Ans.*

3. An aircraft in level flight is said to make a standard turn when it makes a complete circular turn in two minutes. (a) What is the banking angle in a standard turn if the velocity of the aircraft is 432 kph ? (b) What is the radius of the circle in which it turns.

Sol. The forces acting on the aircraft are : (i) mg, downwards, (ii) L, lift of air at an angle θ with the vertical.

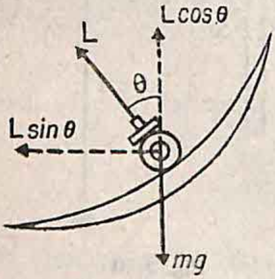


Fig. 5.12

Resolving along the horizontal and vertical,

$$L\cos\theta = mg$$

and $$L\sin\theta = \frac{mv^2}{r}.$$

Dividing, $$\tan\theta = \frac{v^2}{rg}.$$

Here $v = 432 \text{ kph} = \dfrac{432 \times 1000}{3600}$

$= 120$ m/s.

$$\omega = 2\pi n = 2\pi \times \frac{2}{120} = \frac{4\pi}{120} = \frac{\pi}{30}.$$

$$v = \omega r.$$

$$\therefore \quad r = \frac{120 \times 30}{\pi} = \frac{3600}{\pi} \text{m} = 1146 \text{ m.} \quad \textit{Ans.}$$

$$\therefore \quad \tan\theta = \frac{120 \times 120 \times \pi}{3600 \times 9\cdot8} = 1\cdot282 \text{ or } \theta = 52^\circ. \quad \textit{Ans.}$$

*4. A 4-kg block is attached to a vertical rod by means of two strings of equal length. When the system rotates about the axis of the rod the strings are extended as shown in the diagram. How many revolutions per minute must the system make in order that the tension in the upper cord shall be 20 kgf ? (b) What is then the tension in the lower cord ?

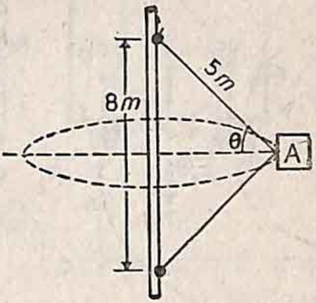


Fig. 5.13

Sol. The forces acting on the block (i) mg, downward, (iii) T_1, tension in the upper cord at inclination θ with the horizontal, (iii) T_2, tension in the lower cord at inclination θ to the horizontal on the other side.

Resolving along the horizontal and vertical we have

$$T_1 \cos\theta + T_2 \cos\theta = m\omega^2 r = m\omega^2 l \cos\theta \quad (\because \; r = l\cos\theta)$$

or $$T_1 + T_2 = m\omega^2 l, \qquad \text{.. (i)}$$

and $T_1 \sin\theta = T_2 \sin\theta + mg$ ($\because$ there is no vertical acceleration)

or $$(T_1 - T_2) \sin\theta = mg. \qquad \text{.. (ii)}$$

Using the given data in Eq. (i),

$$20 \times g + T_2 = 4\omega^2 \times 5$$

or $$20g + T_2 = 20\omega^2$$

and in Eq. (ii), $$(20g - T_2) \times \tfrac{4}{5} = 4 \times g$$

or $$20g - T_2 = 5g.$$

Adding, $$40g = 20\omega^2 + 5g$$

$$\omega^2 = \frac{359}{20} \quad \text{or} \quad \omega = \sqrt{\frac{35g}{20}}$$

or $$n = \frac{1}{2\pi}\sqrt{\frac{7g}{4}} = \cdot 33 \text{ per sec} = 19{\cdot}8 \text{ per min.} \quad \textit{Ans.}$$

$$T_2 = 20g - 5g = 15g = 15 \text{ kgf.} \quad \textit{Ans.}$$

*5. Show that the vertical reaction on the outer wheel of a car moving at a speed along an unbanked curved path is always greater than that on the inner wheel and at very high speeds it will overturn about the outer wheel away from the centre. Assume sufficient friction to prevent skidding.

Sol. Let a car of mass m move in a circular track (unbanked) of radius r. Let $2u$ be the distance between inner and outer wheels and h is the height of the cg of the car above the ground. Let N_1 and N_2 be the normal reactions on the inner and outer wheels and F_1 and F_2 are the frictional forces on them towards the centre of the track respectively.

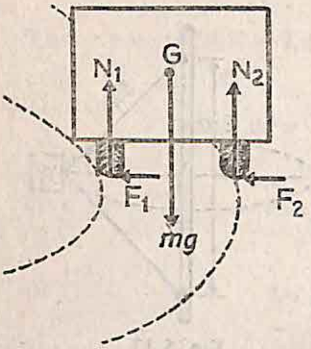


Fig. 5.14

For rotational equilibrium we have

$$F_1+F_2=\frac{mv^2}{r}$$

$$N_1+N_2=mg.$$

Taking moments about the cg of the car, we have for no turning of the car

$$(F_1+F_2)\,h+N_1\,a=N_2\,a.$$

Thus $\quad N_1+N_2=mg \quad$ and $\quad N_2-N_1=\dfrac{mv^2h}{r.a}$

Hence $\quad N_1=\frac{1}{2}\,m\left(g-\dfrac{v^2h}{r.a}\right)$ and $N_2=\frac{1}{2}\,m\left(g+\dfrac{v^2h}{r.a}\right)$.

Clearly $\quad N_2>N_1$. Proved.

As the speed is increased, N_1 decreases and at speed $v=\sqrt{\dfrac{g\,r.a}{h}}$ it vanishes altogether. That means the inner wheel loses contact with the ground and the car tends to turn about the outer wheel away from the centre.

Motion in a Vertical Plane

*6. A stone of mass $\frac{1}{2}$ kg is attached to a string of length 50 cm which will break if the tension in it exceeds 20 N. The stone is whirled in a vertical circle, the axis of rotation being at a height 1 m above the ground. The angular speed slowly increases, until the string breaks. In what position is this most likely to occur and at what angular speed ? Where will the stone hit the ground ?

Sol. Let us consider any position of the stone. The forces acting are : (i) mg, downwards and (ii) T, tension towards the centre. Resolving the forces along the string towards the centre and perpendicular to it, we have from 'dynamics of circular motion',

$$T-mg\cos\theta=\frac{mv^2}{l}$$

where l is the length of the string

and $$mg\sin\theta = ma_\tau$$
where a_τ = tangential acceleration.

From energy conservation we have
$$mgd+\tfrac{1}{2}\,mv_0^2 = mgd+mg(l-l\cos\theta)+\tfrac{1}{2}mv^2$$
where v_0 is the speed at the lowest position and d is the height above the ground of the lowest position.

or $$v_0^2 = v^2+2gl(1-\cos\theta)$$

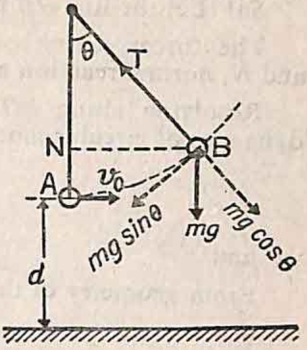


Fig. 5.15

$$\therefore\quad T-mg\cos\theta = \frac{m}{l}[v_0^2-2gl(1-\cos\theta)]$$

$$=\frac{mv_0^2}{l}-2mg(1-6\theta)$$

or $$T=\frac{mv_0^2}{l}-2mg+3\,mg\cos\theta.$$

Obviously, T is maximum when $\cos\theta$ is maximum.

The string is likely to break where the tension is maximum, that is, when $\cos\theta=1$ or $\theta=0°$.

Hence the string is likely to break at the lowest poistion.

$$T_{max}=\frac{mv_0^2}{l}+mg.$$

$$\therefore\quad 20=\frac{\frac{1}{2}\times v_0^2}{\frac{1}{2}}+\tfrac{1}{2}\times 9\,8 \quad\text{or}\quad v_0^2=15\cdot1 \quad\text{or}\quad v_0=3\cdot88 \text{ m/s}.$$

$$\because\quad v=\omega r,\quad \therefore\quad \omega=\frac{v_0}{l}=\frac{3\cdot88}{\frac{1}{2}}=7\cdot76 \text{ rad/s.} \quad \textit{Ans.}$$

After the string breaks, the stone is thrown horizontally.

$$\therefore\quad 1=0,\ t+\tfrac{1}{2}\times 9\cdot8t^2 \quad\text{or}\quad t=\sqrt{\frac{2}{9\cdot8}}.$$

$$x \text{ (horizontal distance)} = v_0t = 3\cdot88\times\sqrt{\frac{2}{9\cdot8}}=1\cdot75 \text{ m. } \textit{Ans.}$$

*7. A loop-the-loop apparatus is made of a quadrant of radius $4r$ joined to a semicircle of radius r as shown. The track is frictionless and a small particle of mass m slides down starting from the highest point A of the quadrant. Calculate the speed, tangential and normal acceleration of the particle at B. With what velocity does it leave the track at D ?

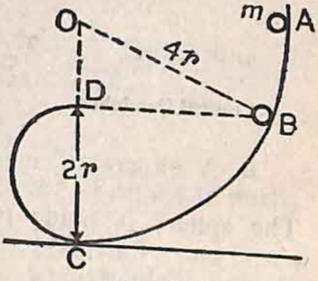


Fig. 5.16

Sol. Let the line OB make an angle θ with the horizontal line DB.

The forces acting on the particle at B are : mg, downwards and N, normal reaction along BO.

Resolving along BO and perpendicular to it we have from dynamics of circular motion,

$$N - mg\sin\theta = \frac{mv^2}{4r}$$

and $$mg\cos\theta = ma_\tau.$$

From geometry of the figure,

$$\sin\theta = \frac{2r}{4r} = \tfrac{1}{2}$$

$$\therefore \quad \theta = 30^\circ.$$

$$\therefore \quad N - \tfrac{1}{2}mg = \frac{mv^2}{4r}$$

and $$g\cos 30^\circ = a_\tau$$

$$\therefore \quad a_\tau = \frac{g\sqrt{3}}{2}.$$

From 'energy conservation',

$$\text{energy at } A = \text{energy at } B$$

or $$mg.4r = mg.2r + \tfrac{1}{2}mv^2$$

or $$v^2 = 4gr \quad \text{or} \quad v = 2\sqrt{gr}$$

$$a_r \text{ (radial acceleration)} = \frac{v^2}{4r} = \frac{4gr}{4r} = g.$$

From 'energy conservation' between A and C,

$$mg.4r = mg.2r + \tfrac{1}{2}mv_c^2$$

or $$v_c^2 = 4gr \quad \text{or} \quad v_c = 2\sqrt{gr}.$$

Thus $v = 2\sqrt{gr}$, $a_r = g$ and $a_\tau = \dfrac{g\sqrt{3}}{2}$ at B. *Ans.*

and at C, $v_c = 2\sqrt{gr}$. *Ans.*

Conical Pendulum

8. A shpere of mass $m = 200$ gm is attached to an inextensible string of length $l = 130$ cm whose upper end is fixed to the ceiling. The sphere is made to describe a horizontal circle of radius $r = 50$ cm. Calculate the time of one revolution. What is the tension of the string ? **(I. I. T. 1974)**

Sol. The forces acting on the sphere are : (i) mg, downwards and (ii) T, tension. Resolving along horizontal and vertical direction, we have from the dynamics of circular motion.

$T\sin\theta = m\,\omega^2 l \sin\theta \quad (\because \; r = l\sin\theta)$

and $T\cos\theta = mg$ (there is no acceleration along the vertical).

$T = m\omega^2 l$ and $T\cos\theta = mg$.

From the geometry of the figure,

$$\sin\theta = \frac{r}{l}. \quad \text{and} \quad \cos\theta = \frac{\sqrt{l^2 - r^2}}{l}$$

$$\therefore \quad T = m\omega^2 l \quad \text{and} \quad \frac{T.\sqrt{l^2 - r^2}}{l} = mg$$

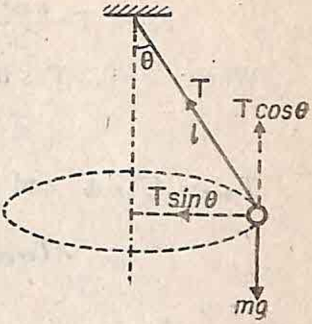


Fig. 5.17

$$\text{Dividing,} \quad \omega^2 = \frac{g}{\sqrt{l^2 - r^2}} \quad \text{or} \quad \omega = \left(\frac{g^2}{l^2 - r^2}\right)^{\frac{1}{4}}$$

$$\text{and} \quad T = m\left(\frac{l^2 g^2}{l^2 - r^2}\right)^{\frac{1}{2}}$$

$$\therefore \quad \omega = \left(\frac{9{\cdot}8^2}{1{\cdot}3^2 - {\cdot}5^2}\right)^{\frac{1}{4}} = 2{\cdot}86$$

$$\text{or} \quad T = \frac{2\pi}{\omega} = \frac{2\pi}{2{\cdot}86} = 2{\cdot}2\text{s.} \quad \textit{Ans.}$$

$$T = {\cdot}2\left(\frac{1{\cdot}3^2 \times 9{\cdot}8^2}{1{\cdot}3^2 - {\cdot}5^2}\right)^{\frac{1}{2}} = 2{\cdot}12\ N. \quad \textit{Ans.}$$

*9. A horizontal rod of length 1 m is rotated about a vertical axis passing through one of its ends. What is the number of rps at which this rod ruptures ? Breaking stress of copper $= 3 \times 10^8$ Nm^{-2}, density of copper $= 8900$ kgm^{-3}.

Sol. Consider an element of the rod at a distance x from the axis and width dx. Let T and $T \mp dT$ be the tensions at the ends of the element. Centripetal force on the element $= -dT = \left(\frac{m}{l}dx\right)\omega^2 x$ where m is the mass of the rod.

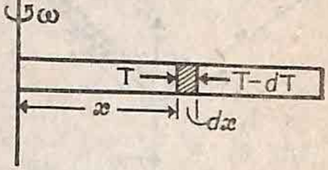


Fig. 5.18

$$\text{Integrating,} \quad -T = \frac{m\omega^2}{l} \cdot \frac{x^2}{2} + c.$$

$$\text{When} \quad x = l,\ T = 0; \quad \therefore \quad 0 = \frac{m\omega^2}{l} \cdot \frac{l^2}{2} + c$$

$$\therefore\quad -T=\frac{m\omega^2}{2l}(x^2-l^2)\quad\text{or}\quad T=\frac{m\omega^2}{2l}(l^2-x^2)$$

When $x=0$, T is maximum.

$$\therefore\quad T_{max}=\frac{m\omega^2}{2l}l^2=\tfrac{1}{2}m\omega^2 l$$

$T_{max}=AT_{break}$ and $m=Al\rho$ where $A=$ area of cross-section, $T_{break}=$ breaking stress

$$\therefore\quad AT_{break}=\tfrac{1}{2}Al\rho\omega^2 l\quad\text{or}\quad T_{break}=\tfrac{1}{2}\rho\omega^2 l^2$$

or

$$\omega=\sqrt{\frac{2T_{break}}{\rho l^2}}$$

$$\therefore\quad \omega=\sqrt{\frac{2\times 3\times 10^8}{8900\times 1^2}}=259{\cdot}6$$

or $n=41{\cdot}3$ per sec. *Ans.*

10. A device (see figure) consists of as a smooth L-shaped rod located in a horizontal plane and a sleeve of mass m attached by a weightless spring to a point B. The spring stiffness is k. The whole system rotates with a constant angular velocity ω about a vertical axis passing through the point O. Find the elongation of the spring if its natural length is l_0. How is the result is affected by the rotation direction ?

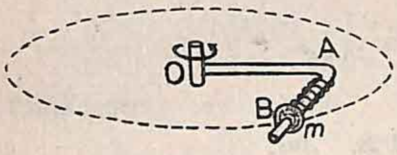


Fig. 5.19

Sol. The forces acting on the sleeve are : (i) mg, downwards, (ii) N, normal reaction of the rod perpendicular to it in the plane of rotation, (iii) R, vertical upward reaction of the rod and (iv) the force of the spring, F, along the rod.

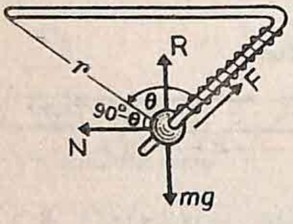


Fig. 5.20

Resolving all the forces along the normal and tangent we have

$$F\cos\theta+N\sin\theta=\frac{m\omega^2(l_0+\triangle l)}{\cos\theta}$$

$$(\because\quad r\cos\theta=l_0+\triangle l)$$

$F\sin\theta=N\cos\theta$ (there is no tangential acceleration as the velocity is uniform)

and $R=mg$ (as there is no vertical acceleration)

Eliminating N between the first two equations

$$F\cos\theta+\frac{F\sin\theta}{\cos\theta}.\sin\theta=\frac{m\omega^2(l_0+\triangle l)}{\cos\theta}$$

or $$F = m\omega^2(l_0 + \triangle l).$$

But $$F = k\triangle l$$

$$\therefore \quad k\triangle l = m\omega^2 l_0 + m\omega^2 \triangle l$$

or $$\triangle l = \frac{m\omega^2 l_0}{k - m\omega^2}. \quad Ans.$$

*11. A closed chain of mass m is attached to a vertical rotating shaft by means of a thread (see figure), and rotates with a constant angular velocity ω. The thread forms an angle θ with the vertical. Find the distance between the chain's centre of gravity and the rotation axis, and the tension of the thread.

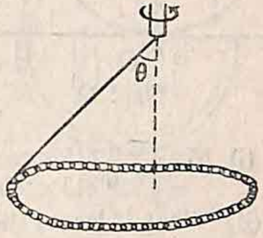


Fig. 5.21

Sol. Let us solve this problem by D'Alemberts principle. We can take the chain to the stationary in the position shown in the figure by considering an additional 'inertia force' at the c.g. of the chain. The c.g. of the chain moves in a circle of radius r about the axis of the shaft. The real forces acting on the chain are : (i) mg, at the c.g. of the chain vertically downwards, (ii) T, tension of the string. In addition to these two forces, let us apply the 'inertia force', that is, 'centrifugal force' $m\omega^2 r$ at the c.g. of the chain. Now the chain may be treated to be at rest and the sought quantities can be found from conditions for 'static equilibrium' of the chain. Resolving forces along the vertical and horizontal we have

$$T\cos\theta = mg$$

and $$T.\sin\theta = m\omega^2 r.$$

$$\therefore \quad T = \frac{mg}{\cos\theta}$$

and $$r = \frac{T\sin\theta}{m\omega^2} = \frac{g\tan\theta}{\omega^2}. \quad Ans.$$

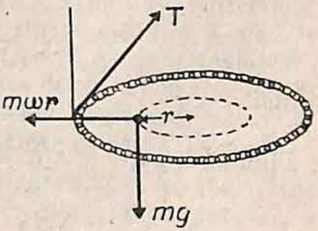


Fig. 5.22

*12. A block of mass $\frac{1}{2}$ kg is placed on the highest point of a fixed smooth sphere of radius $\frac{1}{2}$ m. The body is gently pushed to one side. Calculate : (i) the force exerted by the block on the sphere when it makes an angle 30° with the vertical. (ii) At what angle does it leave the sphere ? (iii) At what distance from the point of contact of the sphere does the block hit the ground ?

Sol. Let θ be the angular displacement with the vertical. The forces acting are : (i) mg, downwards, (ii) N, normal reaction of the

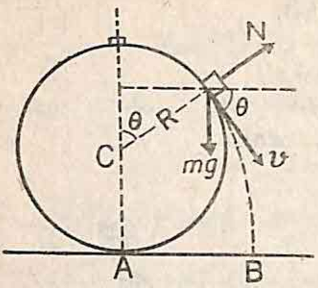


Fig. 5.23

sphere away from the centre. Resolving force along the inward normal and tangent we have

$$mg\cos\theta - N = ma_\tau = m\frac{v^2}{R} \quad .. \quad \text{(i)}$$

and $\quad mg\sin\theta = ma_\tau. \quad .. \quad$ (ii)

From energy conservation (centre as reference)

$$mgR = mgR\cos\theta + \tfrac{1}{2}mv^2 \quad .. \quad \text{(iii)}$$

Eliminating v^2 between (i) and (iii)

$$mg\cos\theta - N = 2mg(1-\cos\theta)$$

or $\quad N = mg(3\cos\theta - 2).$

(i) $N = \frac{1}{2}\times 9{\cdot}8(3\times\cos 30^\circ - 2)$
$= \frac{1}{2}\times 9{\cdot}8(3\times{\cdot}866 - 2) = 2{\cdot}9$ N. *Ans.*

(ii) When it leaves the sphere $N=0$.

$$\therefore \quad 3\cos\theta - 2 = 0$$

$$\theta = \cos^{-1}\tfrac{2}{3} = 48^\circ 11'. \quad \textit{Ans.}$$

(iii) Putting $N=0$ and $\cos\theta = \frac{2}{3}$ in (i) we have,

$$\tfrac{2}{3}g = \tfrac{1}{2}.\,\frac{v^2}{\frac{1}{2}}$$

or

$$v^2 = \tfrac{2}{3}g.$$

Now the block becomes a projectile of initial velocity $\sqrt{\frac{2}{3}g}$ at inclination $\theta = \cos^{-1}\frac{2}{3}$ with the horizontal at a height

$$R + R\cos\theta = R + \tfrac{2}{3}R = \frac{5R}{3}.$$

$$\therefore \quad \frac{5R}{3} = v\sin\theta t + \tfrac{1}{2}gt^2$$

and

$$x = v\cos\theta t.$$

Eliminating t, $\quad \dfrac{5R}{3} = x\tan\theta + \tfrac{1}{2}g\,\dfrac{x^2}{v^2\cos^2\theta}$

or $\quad \dfrac{5\times\frac{1}{2}}{3} = x\dfrac{\sqrt{5}}{2} + \tfrac{1}{2}g\dfrac{x^2}{\frac{2}{3}g.\frac{4}{9}}.$

$$\frac{5}{6} = \frac{\sqrt{5}x}{2} + \frac{27}{16}x^2$$

or

$$81x^2 + 24\sqrt{5}x - 40 = 0$$

$$x = {\cdot}44 \text{ m.} \quad \textit{Ans.}$$

13. A metallic chain with a length $l = 62{\cdot}8$ cm and whose ends are joined together is fitted onto a wooden disc. The disc rotates

with a speed of $n=60$ revolutions per second. Find the tension of the chain if its mass $m=40$ gm.

Sol. Consider an element AB subtending $2\triangle\theta$ at the centre. The forces acting on it are the tensions at the two ends tangentially in opposite directions as shown in the figure. Resolving along inward normal and perpendicular to it we have.

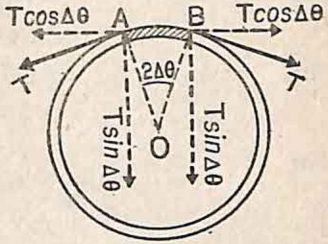


Fig. 5.24

Centripetal force on the element

$$=2T\sin\triangle\theta$$
$$=T.2\triangle\theta.$$

Mass of the element

$$=\left(\frac{m}{l}\right).2\triangle\theta\times R$$

where R is the radius of the disc.

Now $l=2\pi R.$

$$\therefore \quad \text{Mass of the element}=\frac{m}{l}.2\triangle\theta\times\frac{l}{2\pi}=\frac{m\triangle\theta}{\pi}$$

$$\therefore \quad T.2\triangle\theta=\left(\frac{m\triangle\theta}{\pi}\right)\times\omega^2\times\frac{l}{2\pi}.$$

$$\therefore \quad T=\frac{m\omega^2 l}{4\pi^2}=mn^2l \quad (\because \quad \omega^2=4\pi^2n^2)$$

$$\therefore \quad T=\cdot 040\times 60^2\times\cdot 628=9\cdot 2 \text{ kgf.} \quad \textit{Ans.}$$

*14. A spaceship of mass m_0 moves in the absence of external forces with a constant velocity v_0. To change the direction of motion, a jet engine is switched on. It starts ejecting a gas with velocity u relative to the spaceship and directed at right angles to the spaceship's motion. The jet engine is shut down when the mass of the spaceship decreases to m. Through what angle did the motion of the spaceship deviate due to the jet engine operation ?

Sol. Since the forces is always at right angles to the velocity (a characteristic property of circular motion), the spaceship will move in a circle with constant speed v_0 but with variable angular speed ω because the mass of the spaceship changes. The force being perpendicular to the velocity, there is no change in the magnitude of velocity. Let r be the instantaneous radius and ω is the instantaneous angular speed. Then $v_0=\omega r$.

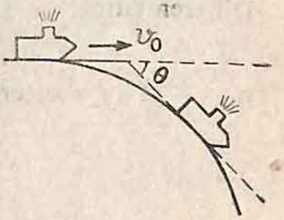


Fig. 5.25

The thrust on the spaceship $=u\,\dfrac{dm}{dt}$.

By dynamics of circular motion

$$u\frac{dm}{dt}=m\omega^2 r.$$

or

$$u\frac{dm}{d\theta}\cdot\frac{d\theta}{dt}=m\omega^2 r$$

or

$$u\frac{dm}{d\theta}\omega=m\omega^2 r \qquad \left(\because \quad \omega=\frac{d\theta}{dt}\right)$$

or

$$u\frac{dm}{d\theta}=m\omega r$$

$$=mv_0.$$

$$\therefore \quad \frac{u}{v_0}\cdot\frac{dm}{m}=d\theta.$$

$$\therefore \quad \frac{u}{v_0}\int_{m_0}^{m}\frac{dm}{m}=\int_0^{\alpha} d\theta$$

$$\theta=\frac{u}{v_0}\log\frac{m}{m_0}. \quad \textit{Ans.}$$

Simple Harmonic Motion

1. A spring of force constant 'k' carries a mass m in a field free space. Calculate the time period of oscillation when the mass is displaced a little and then released. Do it by 'energy conservation principle' and also by consideration of dynamics of the motion.

Sol. When the spring is extended by x, energy of the spring in $\frac{1}{2}kx^2$ and kinetic energy of the mass is $\frac{1}{2}mv^2$.

$\therefore$ Energy of the system (spring+mass)$=\frac{1}{2}kx^2+\frac{1}{2}mv^2$ (neglecting the kinetic energy of the spring).

This is a constant by the energy conservation principle.

$$\therefore \quad \tfrac{1}{2}kx^2+\tfrac{1}{2}mv^2=c \text{ (a constant)}$$

Differentiating, $\quad \frac{1}{2}k2x.\frac{dx}{dt}+\frac{1}{2}m2v.\frac{dv}{dt}=0.$

But $\frac{dv}{dt}=f$, acceleration and $\frac{dx}{dt}=v$, velocity.

$$\therefore \quad \tfrac{1}{2}k2x.v+\tfrac{1}{2}m2v.f+0$$

$$f=-\frac{k}{m}x.$$

$\therefore \quad f \propto -x$. So the motion is simple harmonic.

$\therefore \quad \omega^2 = \frac{k}{m}$, or $T = 2\pi\sqrt{\frac{m}{k}}$ and n (frequency) $= \frac{1}{2\pi}\sqrt{\frac{k}{m}}$.

By 'dynamics': When the spring is extended by x, the force called into play by it on the body is kx in the direction opposite to x.

$$\therefore \quad -kx = m \times f$$

or
$$f = -\frac{k}{m}x.$$

$\therefore$ Motion is simple harmonic and $\omega^2 = \frac{k}{m}$.

$$\therefore \quad T = 2\pi\sqrt{\frac{m}{k}} \quad \text{and} \quad n = \frac{1}{2\pi}\sqrt{\frac{k}{m}}.$$

Note. Though here the deduction from dynamics is much easier, the energy conservation principle is helpful when the spring is not light.

2. Find the period of small oscillations in a vertical plane performed by a ball of mass m fixed at the middle of a horizontally stretched string of length l. The tension of the string is assumed to be constant and equal to $F \gg mg$.

Sol. Let the string make angle α with the horizontal position in the equilibrium position.

For equilibrium we have

$2F\sin\alpha = mg \quad .. \quad$ (i).

Let us consider an instantaneous small angular displacement by θ.

The unbalanced upward force

$= 2F\sin(\alpha+\theta) - mg$

$= 2F(\sin\alpha.\cos\theta + \cos\alpha.\sin\theta) - mg$

$= 2F\sin\alpha + 2F\cos\alpha\theta - mg$

$(\because \quad \cos\theta = 1, \sin\theta = \theta)$

$= 2F\cos\alpha\theta = 2F\theta \quad (\because \quad \cos\alpha \simeq 1)$

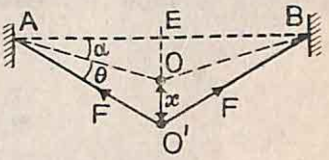


Fig. 5.26

$$= 2F\frac{x}{l/2} \quad \left(\because \quad \theta = \frac{x}{l/2}\right)$$

$$\therefore \quad -4F.\frac{x}{l} = mf$$

$$f = -\frac{4F}{ml} \times x$$

$$\therefore\quad \omega^2=\frac{4F}{ml},\quad \text{or}\quad T=2\pi\sqrt{\frac{ml}{4F}}=\pi\sqrt{\frac{ml}{F}}.\quad \textit{Ans.}$$

3. A simple pendulum of length l is kept immersed in a liquid of density n times less than the density of spherical bob of the pendulum. Calculate the time period of small oscillations of the pendulum.

Sol. Consider a small displacement of the pendulum from the mean position by θ in the anticlockwise direction.

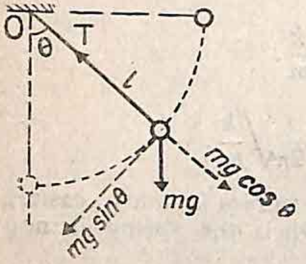


Fig. 5.27

The unbalanced restoring force

$$=\frac{4\pi}{3}r^3\rho g\sin\theta-\frac{4\pi}{3}r^3\sigma g\sin\theta$$

where r = radius of the bob, ρ = density of bob, σ = density of liquid

$$=\frac{4\pi}{3}r^3(\rho-\sigma)g\sin\theta$$

$$=\frac{4\pi}{3}r^3(\rho-\sigma)g.\theta\quad(\because\ \theta\ \text{small})$$

$$=\frac{4\pi}{3}r^3(\rho-\sigma)g.\frac{x}{l}.$$

$$\therefore\quad -\frac{4\pi}{3}r^3(\rho-\sigma)g\,\frac{x}{l}=\frac{4\pi}{3}r^3\rho\times f$$

or

$$f=-\left(1-\frac{\sigma}{\rho}\right)\frac{g}{l}x$$

$$=-\left(1-\frac{1}{n}\right)\frac{g}{l}x\quad\left(\because\ \sigma=\frac{\rho}{n}\right)$$

Therefore, $f\propto -x$ and motion is simple harmonic.

$$\therefore\quad \omega^2=\frac{n-1}{n}\cdot\frac{g}{l},$$

or

$$T=2\pi\sqrt{\frac{nl}{(n-1)g}}=2\pi\sqrt{\frac{l}{\left(1-\frac{1}{n}\right)g}}.\quad \textit{Ans.}$$

*4. A body of mass m fell from a height h onto the pan of a spring balance. The mass of the pan is M and that of the spring is negligible. The force constant of the spring is k. Having stuck to the pan, the body starts performing harmonic oscillations in the vertical direction. Find the period, amplitude and energy of these oscillations.

Sol. Let v be the velocity immediately after impact.

Then $m\sqrt{2gh} = (M+m)v$, or $v = \dfrac{m}{M+m}\sqrt{2gh}$ .. (i)

The period of oscillation of a spring depends on mass attached and its force constant and it is given by

$$T = 2\pi\sqrt{\frac{m}{k}}.$$

$\therefore$ Here, $T = 2\pi\sqrt{\dfrac{M+m}{k}}$, or $\omega = \sqrt{\dfrac{k}{M+m}}$.

The body vibrates about the equilibrium position.

Let the spring be compressed by x_0 from the *equilibrium position.* Then $(M+m)g = kx_0$

or $$x_0 = \frac{(M+m)g}{k} \quad .. \quad \text{(ii)}$$

Let the spring be compressed by x_0' after impact.

By conservation of energy

$$\tfrac{1}{2}(M+m)v^2 + (M+m)gl_0$$
$$= 0 + (M+m)g(l_0 - x_0') + \tfrac{1}{2}kx_0'^2$$

or $$\tfrac{1}{2}(M+m)\frac{m^2}{(M+m)^2}.2gh$$
$$= -(M+m)gx_0' + \tfrac{1}{2}kx_0'^2$$

or $$\frac{m^2}{M+m}gh + (M+m)gx_0'$$
$$= \tfrac{1}{2}kx_0'^2$$

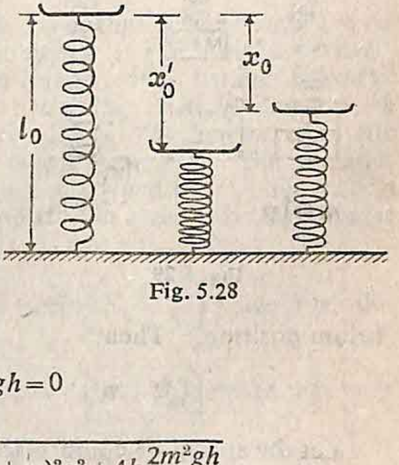


Fig. 5.28

or $$kx_0'^2 - 2(M+m)gx_0' - \frac{2m^2}{M+m}gh = 0$$

$$x_0' = \frac{(M+m)g}{k} \pm \frac{1}{2k}\sqrt{4(M+m)^2g^2 + 4k\frac{2m^2gh}{M+m}}$$

$$= \frac{(M+m)g}{k} \pm \frac{(M+m)g}{k}\sqrt{1 + \frac{2m^2kh}{(M+m)^3g}}.$$

The negative sign is not acceptable because x_0' cannot be negative

$$\therefore \quad x_0' = \frac{(M+m)g}{k} + \frac{(M+m)g}{k}\sqrt{1 + \frac{2m^2kh}{(M+m)^3g}}.$$

Now a (amplitude) $= x_0' - x_0 = \dfrac{(M+m)g}{k}\sqrt{1 + \dfrac{2m^2kh}{(M+m)^3g}}$.

Now, $E=\frac{1}{2}ma^2\omega^2$ (formula)

$$=\frac{1}{2}(M+m)\frac{(M+m)^2g^2}{k^2}\left(1+\frac{2m^2kh}{(M+m)^3g}\right)\times\frac{k}{(M+m)}$$

$$=\frac{(M+m)^2g^2}{2k}\left(1+\frac{2m^2kh}{(M+m)^3g}\right).\ \textit{Ans.}$$

5. A spring of force constant k carries a weight M at the lower end and clamped at the upper end to a rigid fixed support. A bullet of mass m is fired into M vertically upward from below with velocity v. After collision the bullet remains embedded in the mass M. Calculate the period, amplitude and energy of the resulting system.

Sol. Let v' be the velocity immediately after collision.

Then $mv=(M+m)v'$ by momentum conservation.

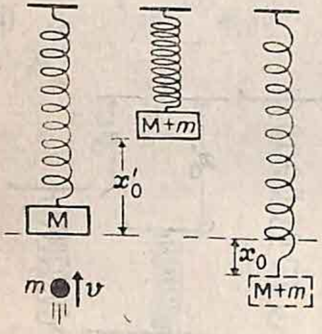


Fig. 5.29

The period of oscillation of a spring loaded with a mass depends only on the mass attached and the force constant of the spring and is given by $T=2\pi\sqrt{\frac{m}{k}}$ (formula).

$$\therefore\quad T=2\pi\sqrt{\frac{M+m}{k}},$$

$$\text{or}\qquad \omega=\sqrt{\frac{k}{M+m}}.$$

The system will oscillate about the equilibrium position. Let the spring be compressed by x_0 in equilibrium position. Then

$$(M+m)g=kx_0,\quad\text{or}\quad x_0=\frac{(M+m)g}{k}.$$

Let the spring be compressed by x_0' after impact.

By energy conservation

$$\frac{1}{2}(M+m)v'^2=(M+m)gx_0'+\frac{1}{2}kx_0'^2$$

$$\text{or}\qquad kx_0'^2+2(M+m)gx_0'-(M+m)\frac{m^2v^2}{(M+m)^2}=0$$

$$\text{or}\qquad kx_0'^2+2(M+m)gx_0'-\frac{m^2v^2}{M+m}=0.$$

$$\therefore\quad x_0'=-2(M+m)g\pm\sqrt{4(M+m)^2g^2+4k.\frac{m^2v^2}{M+m}}\ 2k$$

or $$x_0' = -\frac{(M+m)g}{k} \pm \frac{(M+m)g}{k}\sqrt{1+\frac{m^2v^2}{(M+m)^3g^2}}$$

The negative sign is not acceptable as x_0' can not be negative.

Now $$a \text{ (amplitude)} = x_0 + x_0' \qquad \text{(because } x_0 \text{ is now downward)}$$

or $$a = \frac{(M+m)g}{k}\sqrt{1+\frac{m^2v^2}{(M+m)^3g^2}}. \quad Ans.$$

$\therefore$ E (energy of the resulting SHM) $= \frac{1}{2}ma^2\omega^2$ (formula)

$$= \tfrac{1}{2}(M+m).\frac{(M+m)^2g^2}{k^2}\left(1+\frac{m^2v^2}{(M+m)^3g^2}\right)\times\frac{k}{M+m}$$

$$= \frac{1}{2}\frac{(M+m)^2g^2}{k}\left(1+\frac{m^2v^2}{(M+m)^3g^2}\right). \quad Ans.$$

6. A particle moves along the x-axis according to the law $x = a\cos\omega t$. Find the distance that the particle covers during the time interval from $t=0$ to t.

Sol. According to the law at $t=0$, $x=a$, that is, the particle is at the extreme right end at the initial moment. If t is exactly divisible by $T/4$, that is, $t=n.T/4$ where n is an integer, the distance covered will be na because in every quarter cycle the particle describes a distance by 'a'. If t is not exactly divisible by $T/4$ then we can write $t=n.T/4+t'$ where t' is additional time less than $T/4$. The distance covered now will be na plus the distance described in t'. Now if n is odd, this distance is $na+a\cos\omega t'$ and if n is even this distance is $na+a-a\cos\omega t'$.

Thus $$S = na + a\cos\omega\left(t-\frac{nT}{4}\right) = na + a\cos\left(\omega t - \frac{n\pi}{2}\right) \text{ when } n \text{ is odd}$$

$$= na + a - a\cos\left(\omega t - \frac{n\pi}{2}\right) \text{ when } n \text{ is even}$$

or $$S = a\left[n+1-\cos\left(\omega t-\frac{n\pi}{2}\right)\right] \text{ when } n \text{ is even}$$

$$= a\left[n+\cos\left(\omega t-\frac{n\pi}{2}\right)\right].$$

7. Two cubes with masses m_1 and m_2 were interconnected by a weightless spring of force constant k and placed on a smooth horizontal surface. Then the cubes were drawn closer to each other and released. Find the period of oscillation of the system.

Sol. Each mass will vibrate with the same period.

Since the system is free from external force, the centre of mass of the system will remain stationary. The position of the centre of mass divides the distance in the inverse ratio of masses, that is,

$$l_1 : l_2 = m_2 : m_1$$

and $l_1 + l_2 = l_0$ (natural length of the spring)

$$\therefore \quad l_1 = \frac{l_0}{m_1 + m_2} \times m_2 \quad \text{and} \quad l_2 = \frac{l_0}{m_1 + m_2} \times m_1.$$

Thus, m_1 vibrates due to the portion of the spring of length l_1 and m_2, vibrates due to the remaining portion of the spring.

Since force constant is inversely proportional to the length

$$\frac{k_1}{k} = \frac{l_0}{l_1} k_1 = kl_0 \times \frac{m_1 + m_2}{m_2 l_0} = \frac{k(m_1 + m_2)}{m_2}$$

and
$$k_2 = \frac{k(m_1 + m_2)}{m_1}$$

The period of oscillation of a loaded spring depends on the mass and force constant given by $T = 2\pi \sqrt{\frac{m}{k}}$ (formula).

$$\therefore \quad T_1 = 2\pi \sqrt{\frac{m_1}{k_1}} = 2\pi \sqrt{\frac{m_1}{\frac{k(m_1 + m_2)}{m_2}}} = 2\pi \sqrt{\frac{m_1 m_2}{k(m_1 + m_2)}}$$

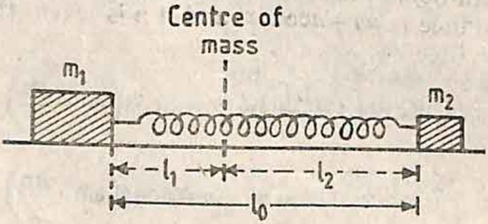


Fig. 5.30

and
$$T_2 = 2\pi \sqrt{\frac{m_2}{k_2}} = 2\pi \sqrt{\frac{m_1 m_2}{k(m_1 + m_2)}}.$$

Thus the two vibrates with the same frequency and it is given by

$$T = 2\pi \sqrt{\frac{m_1 m_2}{k(m_1 + m_2)}}. \quad Ans.$$

or
$$T = 2\pi \sqrt{\frac{\mu}{k}} \quad \text{where} \quad \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}.$$

Note. The formula for time period for two bodies connected by a spring is the same as that of a single body provided mass of the body is replaced by *reduced mass* of the system given by

$$\frac{1}{\mu}=\frac{1}{m_1}+\frac{1}{m_2}.$$

*8. Calculate the frequency of oscillations of a thin uniform rod of mass m and length l hinged at the upper end at O (see figure). The force constants of the springs are k_1 and k_2. The mass of the springs are negligible.

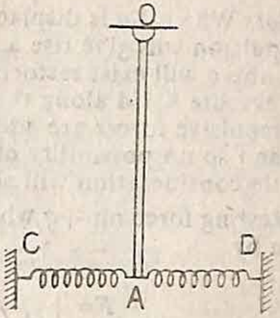


Fig. 5.31

Sol. The rod will perform small angular oscillations about O. Let O be the angular displacement. Let the lower end be displaced to the right by x. The left spring exerts force k_1x to the left and the right hand spring exerts the same force to the left. The weight of the rod 'mg' acts at its centre and that also exerts torque on the rod about O.

$$\tau \text{ (net torque on the rod about } O) = -k_1xl-k_2xl-mg.x/2$$

$$=-(k_1+k_2)xl-mg\frac{x}{2}=-\{(k_1+k_2)l+\tfrac{1}{2}mg\}x$$

$$=-\{(k_1+k_2)l+\tfrac{1}{2}mg\}.\theta l \quad \left(\because \quad \theta=\frac{x}{l}\right)$$

In angular motion

torque = moment of inertia × angular acc.

i.e. $\tau=I\alpha$ (formula in the standard form)

$$\therefore \quad -\{(k_1+k_2)l+\tfrac{1}{2}mg\}l\theta=(\tfrac{1}{3}ml^2)\alpha \quad (\because \quad I=\tfrac{1}{3}ml^2 \text{ for a rod})$$

$$\therefore \quad \alpha=-\left\{\frac{3(k_1+k_2)}{m}+\frac{3g}{2l}\right\}\theta$$

Thus $\alpha \propto -\theta$. Therefore motion is simple harmonic.

$$\therefore \quad \omega^2=\frac{3(k_1+k_2)}{m}+\frac{3g}{2l}.$$

or $$n \text{ (frequency)}=\frac{1}{2\pi}\sqrt{\frac{3(k_1+k_2)}{m}+\frac{3g}{2l}}. \quad \textit{Ans.}$$

*9. Two point charges $+Q$ and $+Q$ are placed $2a$ apart. Show that a point positive charge $+q$ placed at the centre of the line joining the two will execute simple harmonic if slightly displaced along the line and a negative charge $-q$ will perform simple harmonic oscillations along the perpendicular to this line. Calculate the frequency of these oscillations.

Sol. When $+q$ is displaced a little along the line, the two forces of repulsion will give rise a resultant force in the opposite direction. Thus there will exist restoring force on $+q$ along this line. So it may execute SHM along this line. If it is displaced sideways, the two repulsive forces are added to repel $+q$ further away from the line and so no possibility of to and fro oscillation along this line. A little consideration will show that it just the other way for $-q$.

Resting force on $+q$ when slightly displaced to the right by x is given by

$$F=+\frac{1}{4\pi\epsilon_0}\cdot\frac{Qq}{(a+x)^2}-\frac{1}{4\pi\epsilon_0}\cdot\frac{Qq}{(a-x)^2}$$

$$=\frac{1}{4\pi\epsilon_0}\times\frac{Qq}{a^2}\left[\frac{1}{\left(1+\frac{x}{a}\right)^2}-\frac{1}{\left(1-\frac{x}{a}\right)^2}\right]$$

$$=\frac{1}{4\pi\epsilon_0}\cdot\frac{Qq}{a^2}\left[\left(1+\frac{x}{a}\right)^{-2}-\left(1-\frac{x}{a}\right)^{-2}\right]$$

$$F=\frac{1}{4\pi\epsilon_0}\cdot\frac{Qq}{a^2}\left[\left(1-\frac{2x}{a}\right)-\left(1+\frac{2x}{a}\right)\right]$$

(expanding and retaining only significant terms)

$$=-\frac{Qq}{\pi\epsilon_0 a^3}.x$$

Force = mass × acceleration

$$\therefore\quad -\frac{Qq}{\pi\epsilon_0 a^3}x=mf \quad\text{or}\quad f=-\frac{Qq}{\pi\epsilon_0 ma^3}x.$$

∴ Motion is simple harmonic

$$\therefore\quad \omega^2=\frac{Qq}{\pi\epsilon_0 ma^3},\quad\text{or}\quad n=\frac{1}{2\pi}\sqrt{\frac{Qq}{\pi\epsilon_0 ma^3}}.\quad \textit{Ans.}$$

When $-q$ is displaced laterally by x.

Restoring force $= -2F\sin\theta$ where F is the force of attraction due to either change on $-q$.

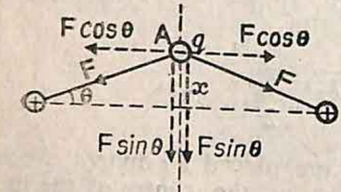


Fig. 5.32

$$=-2.\frac{1}{4\pi\epsilon_0}.\frac{Qq}{a^2+x^2}\cdot\frac{x}{\sqrt{a^2+x^2}}$$

$$=-\frac{Qqx}{2\pi\epsilon_0(a^2+x^2)^{\frac{3}{2}}}$$

$$= -\frac{Qqx}{2\pi\epsilon_0 a^3} \quad \text{(neglecting } x^2 \text{ in comparison to } a^2\text{)}$$

Force = mass × acceleration.

$$\therefore \quad -\frac{Qqx}{2\pi\epsilon_0 a^3} = mf$$

or

$$f = -\frac{Qq}{2\pi\epsilon_0 m a^3} \times x.$$

∴ Motion is simple harmonic.

$$\therefore \quad \omega^2 = \frac{Qq}{2\pi\epsilon_0 m a^3}, \quad \text{or} \quad n = \frac{1}{2\pi}\sqrt{\frac{Qq}{2\pi\epsilon_0 m a^3}}.$$

EXERCISES

1. A particle of mass m moves along a circle of radius R. Find the modulus of the average force acting on the particle over the distance equal to a quarter of the circle if the particle moves (a) uniformly with speed v; (b) with constant tangential acceleration a_τ, the initial acceleration being equal to zero.

2. A small sphere of mass m suspended by a thread is first taken aside so that it forms 90° with the downward vertical and then released. Find

(a) the total acceleration of the sphere and the tension of the thread in terms of θ, the angle made by the thread with the vertical;

(b) the tension in the thread at the moment when the vertical component of the sphere's velocity is maximum;

(c) The angle θ when the total acceleration of the sphere is directed horizontally.

3. A small body suspended by a thread swings in a vertical plane so that its acceleration in the extreme positions and at the lowest positions is the same. Find the angle made by the thread with the vertical in the extreme positions.

*4. A sleeve A can slide freely along a smooth rod bent in the shape of a half circle of radius R. The system is set in rotation with a constant angular velocity ω about a vertical axis 00′. Find the angle θ corresponding to the steady position of the sleeve.

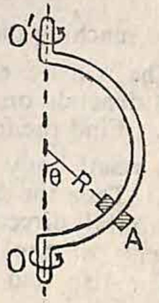


Fig. 5.33

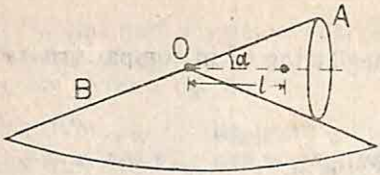


Fig. 5.34

*5. A round cone A of mass m and half angle α rolls uniformly and without slipping along a round conical surface B so that its apex 0 remains stationary (see figure). The centre of gravity of the cone A is at the same level as the point 0 and at a distance l from it. The cones axis moves with angular velocity ω. Find the static frictional force acting on the cone A.

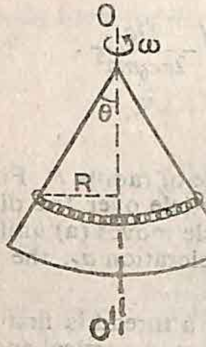


Fig. 5.35

6. A chain of mass m forming a circle of radius R is slipped on a smooth cone with half-angle θ. Find the tension of the chain if it rotates with constant angular velocity ω about a vertical axis (see figure) coinciding with the axis of the cone.

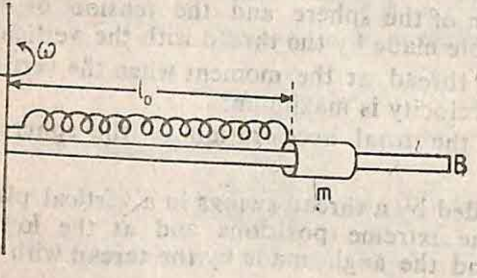


Fig. 5.36

7. A smooth light horizontal rod AB can rotate about a vertical axis passing through its end A. The rod is fitted with a small sleeve of mass m attached to the end A by a weightless spring of length l_0 and stiffness k. What work must be performed to slowly get this system going and reaching the angular velocity ω.

*8. The kinetic energy of a particle moving along a circle of radius R depends on the distance s covered as $K = a s^2$, where a is a constant. Find the force acting on the particle as a function of s.

*9. A small body is placed on a smooth spherical surface of radius R. Then the sphere is imparted a constant acceleration 'a' in the horizontal direction and the body begins sliding down. Find the velocity of the body relative to the sphere at the moment of break-off. Also find the angular displacement with the vertical at the break-off point.

10. A small body A starts sliding from the height h down a smooth inclined groove which passes into a half-circle of radius $h/2$. Find the velocity of the body at the highest point of its trajectory (after breaking off the track).

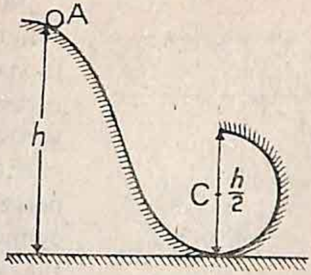


Fig. 5.37

11. A small ball of mass m is suspended by thread of length l. With what minimum velocity has the point of suspension to be shifted in the horizontal direction for the ball to move along the circle about that point ? What will be the tension of the thread at the moment the ball will be passing the horizontal position ?

12. A weightless rod can rotate with respect to the point O in a vertical plane. The rod carries masses m_1 and m_2 at distances r_1 and r_2 from O. The rod is released without any initial velocity from a position forming an angle α with the vertical. Determine the linear velocities of m_1 and m_2 when the rod reaches the vertical position.

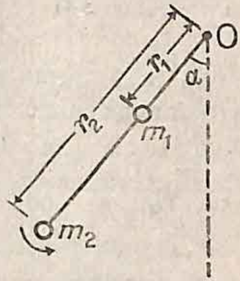


Fig. 5.38

13. A rigid rod AOO' bent at θ to the vertical as shown in the figure rotates with an angular velocity ω about axis OO'. A bead is fitted onto the rod. Find the distance l from the point O at which the qead will be in equilibrium if the coefficient of friction between the bead and the rod is μ.

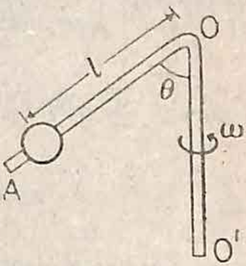


Fig. 5.39

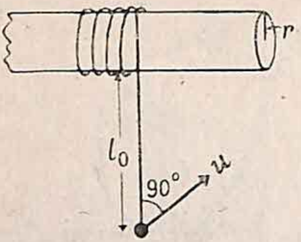


Fig. 5.40

14. A string is attached to a horizontal cylinder of radius $r = 4$ cm around which it passes several times. The free end hangs vertically and supports a particle. The vertical portion is equal to $l_0 = 28$ cm. The particle is given a horizontal velocity u perpendicular (see figure) to the axis of the cylinder and in such a sense as to wind up the string around the rail. What is the value of u so that the string does not slacken in the subsequent motion ?

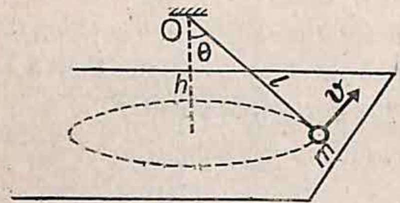


Fig. 5.41

15. A particle of mass m is attached to one end of a light inextensible string of length l. The other end of the string is held fixed at a height h above a smooth horizontal table $(h < l)$. The particle is held on the table with the cord fully extended and projected along the table and at right angles to the string so that it moves in a circle with uniform speed v. Find the maximum velocity for the particle to remain on the table and the corresponding tension.

***16.** To one end of a light inextensible string of length l is attached a particle of mass m resting on a smooth horizontal plane. The string passes through a smooth hole in the table and to its other end is attached a small particle of the same mass. The system is set in rotation with the first particle describing a circle on the table with constant angular velocity ω_1 and the second particle a conical pendulum with constant angular velocity ω_2. Show that the ratio of the lengths of the string above and below the table is $\omega_2^2 : \omega_1^2$. Further show that the motion is possible if $\frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} < \frac{l}{g}$.

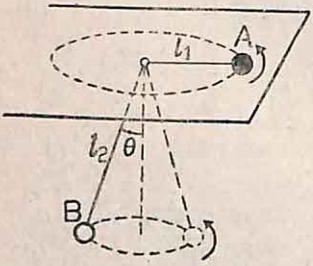


Fig. 5.42

17. Water flows with velocity v along a rubber tube having the form of a ring as shown in the figure. The radius of the ring is R and the radius of the tube $r \ll R$. What force is the rubber tube stretched with?

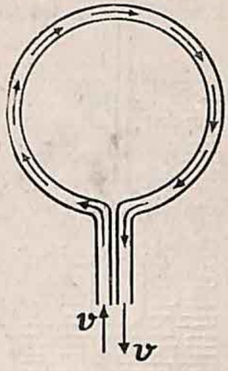


Fig. 5.43

18. A particle of mass $m=100$ gm is suspended from the end of a weightless string of length 1 m and is allowed to swing in a vertical plane. The speed of mass is 2 m/s when it makes an angle of $\theta=60°$ with the vertical. Determine,

(i) the tension of the string at $\theta=60°$ and

(ii) the speed of the particle at the lowest position. ($g=9{\cdot}8$ m/s^2) **(I. I. T. 1971)**

19. A 40 kg mass, hanging at the end of a rope of length l oscillates in a vertical plane with an angular amplitude θ_0. What is the tension in the rope, when it makes θ with the vertical? If the breaking strength of the rope is 80 kg, what is the maximum angular amplitude with which mass can oscillate without the rope breaking? **(I. I. T. 1978)**

Simple Harmonic Motion

1. A particle moves along the x-axis according to the law $x=a\sin\omega t$. Find the distance that the particle covers during the interval from $t=0$ to t.

Find the amplitude, frequency and epoch of the simple harmonic motion represented by $x=3\sin\omega t+4\cos\omega t$.

2. At the moment $t=0$ a particle starts moving along the x-axis according to the law $x=10\sin\pi t$ cm, where t is expressed in seconds. Find the distance that this particle covers during $t=2{\cdot}9$s after the start.

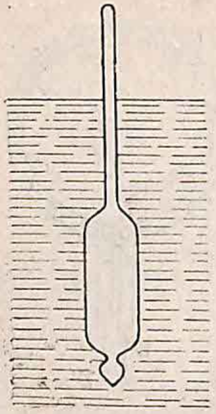


Fig. 5.44

3. Calculate the period of small oscillations of a hydrometer of mass $m = 50$ gm when it is slightly pushed down and then released in water. The radius of the stem of the hydrometer is $r = 3{\cdot}2$ mm.

4. A plank with a bar on it performs horizontal harmonic oscillations with amplitude $a = 10$ cm. Find the coefficient of friction between the bar and the plank if the former starts sliding along the plank when the period of oscillation of the plank becomes equal to $T = 1$ s.

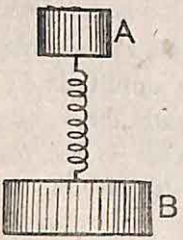


Fig. 5.45

5. A body A of mass $m_1 = 1$ kg and a body B of mass $m_2 = 4$ kg are interconnected by a spring (see figure). The body A performs free oscillations up and down with amplitude $a = 1{\cdot}6$ cm and cyclic frequency $\omega = 25$ rad/s. Neglecting the mass of the spring, find the maximum and minimum values of force that this system $(A+B)$ exerts on the ground.

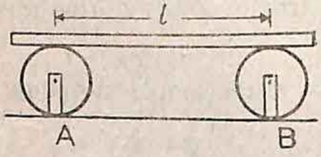


Fig. 5.46

6. A uniform rod is placed on two spinning wheels as shown in the figure. The axes of the wheels are separated by a distance l, the coefficient of friction between the rod and the wheels is k. Show that if the rod is displaced slightly it will perform simple harmonic oscillations. Find the period of these oscillations.

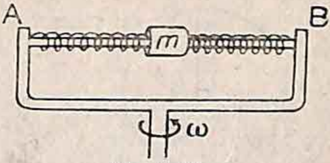


Fig. 5.47

7. In the arrangement shown in the figure the sleeve of mass m is fixed between two identical springs, each of force constant k. The sleeve can slide without friction over horizontal bar AB. The arrangement rotates with a constant angular velocity ω about a vertical axis passing through the middle of the bar. Find the period of small oscillations of the sleeve. At what values of ω will there be no oscillations of the sleeve ?

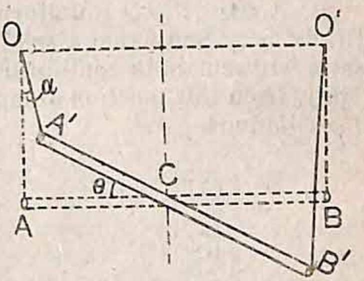


Fig. 5.48

8. A uniform rod of mass m suspended by two identical threads was turned through a small angle about the vertical axis passing through its middle point C. The threads were turned through α in the process. Then the rod was released. Show that the rod performs SHM and find its oscillation period.

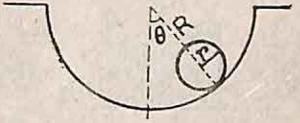


Fig. 5.49

9. A ring of mass m and radius r can roll without slipping over the internal surface of a cylinder of radius R (see figure). Show that motion is simple harmonic if the angle θ is small. Calculate the period of oscillations.

10. If the mass of a spring is not negligible but small compared to the mass M suspended, the period of oscillation is

$$T=2\pi\sqrt{\frac{M+m/3}{k}}.\quad \text{Prove this.}$$

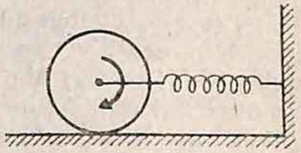


Fig. 5.50

11. A solid cylinder of mass M is attached to a massless spring of force constant k so that the cylinder can roll without slipping along the horizontal surface. Show that the centre of mass of the cylinder executes simple harmonic motion with a period given by

$$T=2\pi\sqrt{\frac{3M}{2k}}.$$

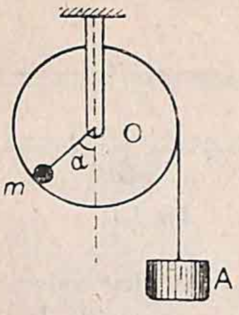


Fig. 5.51

12 A uniform cylindrical pulley of mass M and radius R can rotate freely about the horizontal axis O (see figure). The free end of a thread tightly wound on the pulley carries a weight A. At a certain angle α it counterbalances a point mass m at the rim of the pulley. Find the frequency of small oscillations of the arrangement.

13. A charge $+Q$ is uniformly distributed along a circular ring of radis 'a'. Show that a small negative charge 'q' placed at the centre will remain in equilibrium and it will execute SHM if displaced slightly from this position along the axes. Calculate the frequency of oscillations.

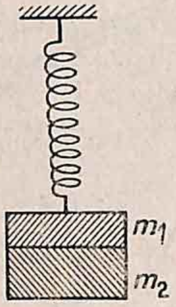


Fig. 5.52

14. Two masses m_1 and m_2 are suspended together by a massless spring of force constant k (see figure). When the masses are in equilibrium, m_1 is removed without disturbing the system. Find the angular frequency and amplitude of oscillation of m_2. **(IIT 1981)**

15. A block performs small oscillations in a vertical plane while moving without friction over the internal surface of a spherical cup. Determine the period of oscillations of the block if the internal radius of the cup is R and the size of the block is negligible in comparison to the radius of the cup.

16. How will the period of oscillations of the block in cup change (cup and block of problem 15) if the cup is placed on a smooth horizontal surface over which it can move without friction ?

17. A string is stretched between two fixed supports $2l$ distance apart with a force f. A point weight of mass m is attached to the string. Determine the period of small oscillations of the weight at right angles to the string. Neglect the force of gravity in comparison to the tension of the string.

TEST YOURSELF

1. Answer the following objective questions : $3 \times 10 = 30$

(i) A motor car is travelling at 30 ms^{-1} on circular road of radius 225 m. It is increasing its speed at the rate of 3 ms^{-2}. The acceleration of the car is (a) 3 ms^{-2}, (b) 4 ms^{-2}, (c) 5 ms^{-2}, (d) 6 ms^{-2}.

(ii) A bottle of soda water is held by the neck and swing rapidly in a vertical circle. The bubbles collect (a) near the bottom, (b) in the middle of the bottle. (c) near the neck, (d) uniformly distributed throughout the volume of the bottle.

(iii) The maximum speed at which a 2500-kg automobile can travel around a curve of radius 1000 m if the road is level and coefficient of friction between the tyres and the road is 0·25 is ($g=10\ \text{ms}^{-2}$) (a) 50 ms^{-1}, (b) 100 ms^{-1}, (c) 30 ms^{-1}, (d) 70 ms^{-1}.

(iv) Assuming that there is sufficient friction to prevent skidding, the tendency to overturn if speed is doubled (a) is increased two-times, (b) is increased four-times, (c) remains unchanged, (d) decreased twice.

(v) A point traversed quarter of a circle of radius R during the interval τ, the modulus of the mean velocity is

(a) $\frac{2R}{\tau}$, (b) $\frac{\sqrt{2}R}{\tau}$, (c) $\frac{R}{\tau}$, (d) $\frac{\pi R}{2\tau}$.

(vi) A particle executes SHM of amplitude 5 cm and period 3s. The velocity of the particle at a distance 4 cm from the mean position (take $\pi=3$) is (a) 8 cm/s, (b) 12 cm/s, (c) 4 cm/s, (d) 6 cm/s.

(vii) The time period of a mass suspended by a spring is T. If the spring is cut into four equal parts and the mass is again suspended by one of the pieces, the period of vibration is now

(a) $\frac{T}{2}$, (b) $\frac{T}{\sqrt{3}}$, (c) $\frac{T}{\sqrt{2}}$, (d) $2T$.

(viii) A particle moves according to the law $x=a\cos\pi t$. The distance covered by it in 2·5 s is

(a) $3a$, (b) $5a$, (c) $2a$, (d) $9a$.

(ix) A coin at a distance 3 cm from the axis of a turntable just flies off when the speed of rotation is ω. The distance at which it will fly off when the speed is increased to 2ω is

(a) ·5 cm, (b) 6 cm, (c) ·75 cm, (d) 7·5 cm.

(x) A lissajous figure has 3 points of contact with a horizontal line and 5 points of contact with vertical line. If the horizontal frequency is 400, the vertical frequency is

(a) 300, (b) 240, (c) 666·7, (d) 500.

2. Explain the following with reasons : $4\times5=20$

(i) Which way is easier to stop a car within a given distance by applying brake or giving a circular turn ?

(ii) Why a force is needed to keep a body move in a circle ?

(iii) Why is the earth flattened along its axis ?

(iv) The resultant of two simple harmonic motions at right angles and of the same frequency is always a circular motion. True or false ? Explain.

(v) Centripetal force is a real force but centrifugal force is a pseudo force. Explain.

Answer any five of the following : $5 \times 10 = 50$

3. A helical spring elongates 10 cm when subjected to a tension of 5 N. Find the mass which should be attached to the bottom of the spring so that when pulled down and released the mass will vibrate twice per second. Find also its maximum velocity when the amplitude of vibration is 1 cm. **(JEE 1979)**

4. In what minimum time after its motion begins will a particle oscillating according to the law $x = 7\sin\frac{1}{2}\pi t$ move from the position of equilibrium to the position $x = -7/2$ units ?

5. A liquid column is kept in a vertical U-tube of uniform cross-section. If the liquid column is depressed a little and left show that the resulting motion is an SHM and find its period in terms of the length of each column.

6. A mass 60 gm is attached to the lower end of a string 4 m long. It revolves along a circular path at a uniform rate in a horizontal plane at 3·2 m vertically below the point of attachment of the upper end of the string. Calculate the tension in the string and the time taken by the mass to describe one complete circle. **(JEE 1981)**

7. A spring of force constant 19·6 N/m hangs vertically. A body of mass 0·2 kg is attached to the free end and released. Assume that the spring was unstretched before the body is released and find how far below the initial position the body descends. Find also the frequency, amplitude and energy of the resulting simple harmonic motion.

8. A small sphere of mass m is attached to a rubber cord and revolves in a horizontal frictionless plane with a constant frequency f. The unstretched length of the cord is l_0. The tension in the cord increases in direct proportion to its elongation, its tension per unit length being k. Find

(a) radius of the uniform circular motion

(b) the tension in the cord.

9. A small ball is suspended from a point A by a thread of length l. A nail is driven into the wall at a distance $l/2$ below A. The ball is drawn aside so that the thread takes up a horizontal position and then released. At what point in the ball's trajectory, will the tension in the thread disappear ?

Full marks 100

Your score....

Answers of Exercises

1. (a) $\frac{2\sqrt{2}mv^2}{\pi R}$; (b) ma_τ.

2. (a) $g\sqrt{1+3\cos^2\theta}$, $T = 3mg\cos\theta$; (b) $T = \sqrt{3}mg$; (c) 54·7°.

3. 53°. **4.** $\cos^{-1}\frac{g}{\omega^2 R}$. **5.** $f = mg\sin\alpha + \frac{m\omega^2 l}{\cos\alpha}$.

6. $T = \frac{mg}{2\pi}\left(\cot\theta + \frac{\omega^2 R}{g}\right)$. **7.** $W = \frac{kl_0^2 n(1+n)}{2(1-n)^2}$ where $n = \frac{m\omega^2}{k}$.

8. $2as\sqrt{1+\frac{s^2}{R^2}}$. **9.** $v = \sqrt{\frac{2}{3}gR}$; $\theta = \cos^{-1}\frac{2+n\sqrt{5+9n^2}}{3(1+n^2)}$

10. $\frac{2}{3\sqrt{3}}\sqrt{gh}$. **11.** $V_{min} = \sqrt{5gl}$; $T = 3mg$.

12. $v_1 = 2r_1\sin\frac{\alpha}{2}\sqrt{g\frac{m_1r_1+m_2r_2}{m_1r_1^2+m_2r_2^2}}$;

$v_2 = 2r_2\sin\frac{\alpha}{2}\sqrt{\frac{m_1r_1+m_2r_2}{m_1r_1^2+m_2r_2^2}\times g}$.

13. $l = \frac{g}{\omega^2}\cdot\frac{\mu\sin\theta-\cos\theta}{\sin\theta(\mu\cos\theta+\sin\theta)}$. **14.** 3·16 m/s.

15. $v_{max} = \sqrt{\frac{g}{h}(l^2-h^2)}$; $T = \frac{mgl}{h}$. **17.** $T = \pi r^2\rho v^2$.

18. $T = {\cdot}89$ N; $v_0 = 3{\cdot}1$ m/s. **19.** $T = mg(3\cos\theta - 2\cos\theta_0)$; $\theta_0 = 60°$.

Simple Harmonic Motion

1. $S = ma + na\sin\left(\omega t - \frac{n\pi}{2}\right)$ where n is the number of quarter cycles in t; 5, $\frac{\omega}{2\pi}$ and 53°.

2. 59·5 cm. **3.** 2·5 s. **4.** ·4. **5.** 59 N or 39 N

6. $T = \pi\sqrt{\frac{2l}{kg}}$ **7.** $T = \frac{2\pi}{\sqrt{\frac{2k}{m}-\omega^2}}$. **8.** $T = 2\pi\sqrt{\frac{l}{3g}}$.

9. $T = 2\pi\sqrt{\frac{2(R-r)}{g}}$. **11.** $T = 2\pi\sqrt{\frac{3M}{2k}}$.

12. $f = \frac{1}{2\pi}\sqrt{\frac{2mg\cos\alpha}{MR+2mR(1+\sin\alpha)}}$. **13.** $f = \frac{1}{2\pi}\sqrt{\frac{Qq}{4\pi\epsilon_0 a^3 m}}$.

14. $\omega = \sqrt{\frac{k}{m_2}}$; $\frac{m_1}{k}g$. **15.** $T = 2\pi\sqrt{\frac{R}{g}}$. **16.** $T = 2\pi\sqrt{\frac{ml}{2f}}$.

Answers of Test Questions

1. (i) (c), (ii) (c), (iii) (a), (iv) (b), (v) (b),
(vi) (d), (vii) (a), (viii) (b), (ix) (c), (x) (b).

2. See solutions. 3. 310 gm; 12·5 m/s. 4. 2·33 s.

5. $T=2\pi\sqrt{\frac{l}{g}}$. 6. $\tau=3{\cdot}6$ s. 7. ·1 m; ·1 joule.

8. $\frac{kl_0}{k-4\pi^2mf^2}$; $\frac{4\pi^2mf^2kl_0}{k-4\pi^2mf^2}$. 9. 42° above the horizontal.

Solutions to Exercises

1. Let us take the centre of the circle as the origin of the frame of reference and rightward direction as x-axis.

$<\vec{F}>$ (average velocity vector) $=m<\vec{a}>$ where $\vec{a}$ is the acceleration vector.

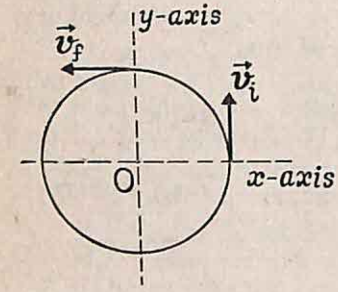


Fig. 5.53

Now $<\vec{a}>=\frac{\vec{v}_f-\vec{v}_i}{\frac{T}{4}}$ where 'i' stands for initial and 'f' for final.

$$\vec{v}_i=v\vec{j} \quad \text{and} \quad \vec{v}_f=-v\vec{i}$$

$$\therefore \quad <\vec{a}>=\frac{4}{T}(-v\vec{i}-v\vec{j})$$

$$\therefore \quad <\vec{F}>=-\frac{4mv}{T}\vec{i}-\frac{4mv}{T}\vec{j}$$

$\therefore$ modulus of the force vector

$$=\sqrt{\left(\frac{4mv}{T}\right)^2+\left(\frac{4mv}{T}\right)^2}=\frac{4\sqrt{2}mv}{T}$$

$$=\frac{4\sqrt{2}mv\omega}{2\pi} \quad \left(\because \quad \omega=\frac{2\pi}{T}\right)$$

$$=\frac{2\sqrt{2}mv^2}{\pi R} \quad (\because \quad v=\omega R)$$

$$\therefore \quad |<\vec{F}>|=\frac{2\sqrt{2}mv^2}{\pi R}. \quad \textit{Ans.}$$

(b) Here $\vec{v}_i=0$ (given)

$$\therefore \quad <\vec{F}>=-\frac{4mv}{T}\vec{i}.$$

$$\therefore \quad |<\vec{F}>| = \frac{4mv}{T}.$$

Now $$a_\tau = \frac{dv}{dt} \text{ (a constant given)}$$

$$\therefore \quad \int dv = \int_0^{T/4} a_\tau dt = a_\tau \frac{T}{4}$$

or $$v = a_\tau . \frac{T}{4}$$

$$\therefore \quad |<\vec{F}>| = \frac{4m}{T} \times \frac{a_\tau T}{4} = ma_\tau. \quad \textit{Ans.}$$

2. Let us take the line through the point of suspension as the reference plane.

(a) By 'conservation of energy'

$0 = mg(-l\cos\theta) + \frac{1}{2}mv^2$

or $2mgl\cos\theta = mv^2$ .. (i).

a_r (centripetal acceleration)

$$= \frac{v^2}{l} = 2g\cos\theta.$$

Tangential force $= mg\sin\theta = ma_\tau$

$\therefore \quad a_\tau = g\sin\theta.$

$\therefore \quad a$ (total acceleration)

$= \sqrt{a_r^2 + a_\tau^2} = g\sqrt{1+3\cos^2\theta}.$ *Ans.*

Centripetal force $= T - mg\cos\theta = m\frac{v^2}{l}.$

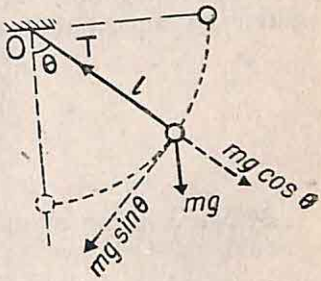


Fig. 5.54

$$\therefore \quad T = mg\cos\theta + 2mg\cos\theta = 3mg\cos\theta. \quad \textit{Ans.}$$

(b) v_y (vertical component of velocity)

$$= v\sin\theta = \sqrt{2gl\cos\theta} \times \sin\theta$$

or $$v_y^2 = 2gl\cos\theta(1-\cos^2\theta) = 2gl(\cos\theta - \cos^3\theta).$$

Differentiating, $$2v_y\frac{dv_y}{d\theta} = 2gl(-\sin\theta + 3\cos^2\theta.\sin\theta).$$

When v_y is maximum, $$\frac{dv_y}{d\theta} = 0.$$

$$\therefore \quad 3\cos^2\theta = 1, \text{ or } \cos\theta = \frac{1}{\sqrt{3}}.$$

∴ When vertical component of velocity is maximum

$$T=3mg\cos\theta=3mg.\frac{1}{\sqrt{3}}=\sqrt{3}mg. \quad Ans.$$

(c) Let α be the angle made by the total acceleration with the thread. Then

$$\tan\alpha=\frac{a_\tau}{a_r}=\frac{g\sin\theta}{2g\cos\theta}=\frac{\tan\theta}{2}.$$

When total acceleration is horizontal

$$\theta+\alpha=90°$$

or

$$\tan\theta=\tan(90-\theta)=\cot\theta.$$

$$\therefore \quad \frac{\tan\theta}{2}=\cot\theta, \quad \text{or} \quad \tan\theta=\sqrt{2}=1{\cdot}414$$

$$\therefore \quad \theta=54{\cdot}7°. \quad Ans.$$

3. Let the thread make angle α. In the extreme position there is no 'centripetal acceleration' but there is tangential acceleration and it is $g\sin\alpha$.

At the lowest position there is no tangential acceleration because there is no tangential force but there is centripetal acceleration and it is $\frac{v^2}{l}$.

$$\therefore \quad g\sin\alpha=\frac{v^2}{l}.$$

By conservation of energy

$$\tfrac{1}{2}mv^2=mgl(1-\cos\alpha)$$

or

$$\frac{v^2}{l}=2g(1-\cos\alpha).$$

$$\therefore \quad g\sin\alpha=2g(1-\cos\alpha)$$

or $\quad \sin\alpha=2(1-\cos\alpha), \quad \text{or} \quad (\sin\alpha-2)^2=4\cos^2\alpha$

or $\quad \sin^2\alpha-4\sin\alpha-4=4-4\sin^2\alpha, \quad \text{or} \quad 5\sin^2\alpha-4\sin\alpha=0$

or $\quad \sin\alpha=\frac{4}{5}={\cdot}8, \quad \text{or} \quad \alpha=53°. \quad Ans.$

4. Let N be normal reaction of the rod on the sleeve.

The radius of the circle on which sleeve moves $=R\sin\theta$.

Centripetal force $=N\sin\theta=m\omega^2R\sin\theta$

or

$$N=m\omega^2R.$$

As there is no vertical acceleration of the sleeve we must have

$$mg=N\cos\theta$$

$$\therefore \quad \frac{mg}{\cos\theta}=m\omega^2R.$$

$$\cos\theta=\frac{g}{\omega^2 R}, \quad \text{or} \quad \theta=\cos^{-1}\frac{g}{\omega^2 R}. \quad \textit{Ans.}$$

5. The forces acting on the cone :

(i) mg, downward,

(ii) N, normal reaction perp. to the surface of B.

(iii) f, frictional force up the plane.

The centre of mass of the cone moves in a circle of radius l.

Resolving all the forces along the horizontal we have, centripetal force on the cone

$$=f\cos\alpha-N\sin\alpha=m\omega^2 l.$$

Since there is no motion of the cone at right angles to the surface of B we have $mg\cos\alpha=N$.

$$\therefore \quad f\cos\alpha-mg\cos\alpha.\sin\alpha=m\omega^2 l.$$

or

$$f=\left(mg\sin\alpha+\frac{m\omega^2 l}{\cos\alpha}\right). \quad \textit{Ans.}$$

6. Consider an element of the chain. Let its length be $\triangle l$ and subtend an angle 2α at the centre.

The mass of the element $=\frac{m}{2\pi}\times 2\alpha=\frac{m\alpha}{\pi}$.

The forces acting on the element :

(i) weight : $\left(\frac{m\alpha}{\pi}\right)g$ downwards,

(ii) reaction of the cone $\triangle N$ on the element at inclination θ,

(iii) tensions T and T tangential to the element in the plane of the ring.

Taking resolved part of all towards the centre O, centripetal force

$$=T\sin\alpha+T\sin\alpha-\triangle N\cos\theta$$
$$=2T\alpha-\triangle N\cos\theta$$
$$(\because \ \alpha \text{ is small, } \sin\alpha=\alpha)$$
$$=\left(\frac{mg}{\pi}\right)\omega^2 R.$$

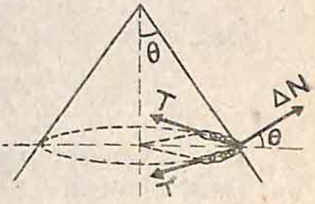


Fig. 5.55

As there is no vertical acceleration,

$$\triangle N\sin\theta=\left(\frac{m\alpha}{\pi}\right)g$$

$$\therefore \quad 2T\alpha-\frac{m\alpha g}{\pi\sin\theta}.\cos\theta=\frac{m\alpha}{\pi}\omega^2 R$$

or
$$2T=\frac{mg}{\pi}\cot\theta+\frac{mg}{\pi}\omega^2R$$
$$=\frac{mg}{\pi}\left(\cot\theta+\frac{\omega^2R}{g}\right)$$
$$\therefore\quad T=\frac{mg}{2\pi}\left(\cot\theta+\frac{\omega^2R}{g}\right).\quad Ans.$$

7. The mass m rotates in a circle of radius l, which is the extended length of the spring.

$\therefore$ Centripetal force on $m=k(l-l_0)$
$$=m\omega^2l$$

or
$$l=\frac{l_0}{1-n}\text{ where }n=\frac{m\omega^2}{k}.$$

W (work done) = change in k.e. of m + energy stored in the spring
$$=\tfrac{1}{2}mv^2+\tfrac{1}{2}k(l-l_0)^2$$
$$=\tfrac{1}{2}ml^2\omega^2+\tfrac{1}{2}k(l-l_0)^2\quad(\because\quad v=l\omega)$$
$$=\tfrac{1}{2}m.\frac{l_0^2\omega^2}{(1-n)^2}+\tfrac{1}{2}k\left(\frac{l_0}{1-n}-l_0\right)^2$$

or
$$W=\tfrac{1}{2}\frac{m\omega^2l_0^2}{(1-n)^2}+\tfrac{1}{2}\frac{kn^2l_0^2}{(1-n)^2}=\tfrac{1}{2}\frac{kl^2}{(1-n)^2}\left[\frac{m\omega^2}{k}+n^2\right]$$

or
$$W=\frac{kl_0^2n(1+n)}{2(1-n)^2}\quad\text{where}\quad n=\frac{m\omega^2}{k}.\quad Ans.$$

8. $F_{radial}=$ centripetal force $=\dfrac{mv^2}{R}$

$$F_{tangential}=ma_\tau=m\frac{dv}{dt}\quad\left(\because\quad a_\tau=\frac{dv}{dt}\right)$$

Here
$$K=as^2.$$
$$\therefore\quad \tfrac{1}{2}mv^2=as^2,\quad\text{or}\quad v^2=\frac{2as^2}{m}.$$

Differentiating,
$$2v\frac{dv}{dt}=\frac{2a}{m}.2s\frac{ds}{dt}$$

or
$$v\frac{dv}{dt}=\frac{2as}{m}.v\quad\left(\because\quad\frac{ds}{dt}=v\right)$$

or
$$\frac{dv}{dt}=\frac{2as}{m}.$$

$$\therefore \quad F_{radial} = m.\frac{2as^2}{m} \Big/ R = \frac{2as^2}{R}$$

$$F_{tangential} = m.\frac{2as}{m} = 2as$$

$$F \text{ (force on the particle)} = \sqrt{F_{rad.}^2 + F_{tang.}^2}$$

or $$F = \sqrt{\frac{4a^2s^4}{R^2} + 4a^2s^2} = 2as\sqrt{1 + \frac{s^2}{R^2}}. \quad Ans.$$

9. By D'Alembert's principle we may assume that the sphere is moving with acceleration a to the right but it is stationary. In place of its motion there is an additional force ma on m to the left. Forces acting on m are : (i) mg, downward,

(ii) N, normal reaction,

(iii) ma, inertia force (leftward).

By 'dynamics of circular motion'

$$mg\cos\theta + ma\sin\theta - N = \frac{mv^2}{R}.$$

At the point of break-off $N = 0$.

$$\therefore \quad g\cos\theta + a\sin\theta = \frac{v^2}{R}$$

or $gR\cos\theta + aR\sin\theta = v^2$. .. (i)

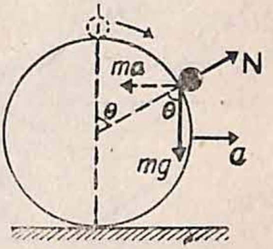


Fig. 5.56

Work done by gravity $= mg(R - R\cos\theta)$

Work done by inertia force

$= -maR\sin\theta$

Work done by $N = 0$.

$\therefore$ Net work done by external forces

$= mgR(1-\cos\theta) - maR\sin\theta$.

By work-energy theorem

$$mgR(1-\cos\theta) - maR\sin\theta = \tfrac{1}{2}mv^2$$

or $$2gR(1-\cos\theta) - 2aR\sin\theta = v^2. \quad .. \text{ (ii)}$$

Eliminating a between (i) and (ii)

$$2gR(1-\cos\theta) + 2gR\cos\theta = 3v^2$$

or $$v^2 = \tfrac{2}{3}gR.$$

Since we have considered the sphere to be at rest, v is here the relative velocity of m relative to the sphere.

$$\therefore \quad v = \sqrt{\tfrac{2}{3}gR}. \quad Ans.$$

Putting the value of v in (i) we have

$$gR\cos\theta + aR\sin\theta = \tfrac{2}{3}gR$$

or $$g\cos\theta + a\sin\theta = \tfrac{2}{3}g, \quad \text{or} \quad \cos\theta + \frac{a}{g}\sin\theta = \tfrac{2}{3}$$

or $\cos\theta + n\sin\theta = \frac{2}{3}$ where $n = \frac{a}{g}$

or $3\cos\theta + 3n\sin\theta = 2$, or $(3n\sin\theta)^2 = (2 - 3\cos\theta)^2$

or $9n^2\sin^2\theta = 4 - 12\cos\theta + 9\cos^2\theta$

or $9n^2 - 9n^2\cos^2\theta = 4 - 12\cos\theta + 9\cos^2\theta$

or $9(1+n^2)\cos^2\theta - 12\cos\theta + 4 - 9n^2 = 0$

or $$\cos\theta = \frac{12 \pm \sqrt{144 - 4 \times 9(1+n^2)(4-9n^2)}}{2 \times 9(1+n^2)}$$

$$\cos\theta = \frac{2 + n\sqrt{5+9n^2}}{3(1+n^2)}.$$

$$\therefore\ \theta = \cos^{-1}\frac{2+n\sqrt{5+9n^2}}{3(1+n^2)}.\quad \textit{Ans.}$$

10. Since the groove is smooth no work is done by friction. Let it break off at an angle θ with the horizontal with speed v.

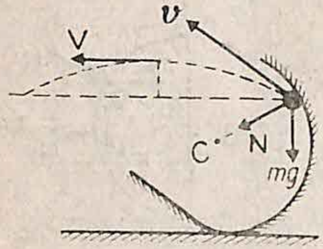


Fig. 5.57

By 'dynamics of circular motion'

$$N + mg\cos\theta = \frac{mv^2}{h/2} = \frac{2mv^2}{h}.$$

At the point of break-off $N = 0$

$$\therefore\quad mg\cos\theta = \frac{2mv^2}{h}$$

or $gh\cos\theta = 2v^2$ .. (i).

By 'energy conservation'

$$mgh = mg\left(\frac{h}{2} + \frac{h}{2}\cos\theta\right) + \tfrac{1}{2}mv^2$$

or $2gh = g(h + h\cos\theta) + v^2$, or $gh(1-\cos\theta) = v^2$ .. (ii).

Combining (ii) and (i) $v = \sqrt{\frac{1}{3}gh}$ and $\cos\theta = \frac{2}{3}$.

Now 'm' becomes a projectile thrown at θ with the horizontal with speed v.

The velocity at the highest point is entirely horizontal and the horizontal component of velocity remains unchanged.

$\therefore$ Velocity at highest point = horizontal component of velocity in the beginning

$$= v\cos\theta = \tfrac{2}{3}\sqrt{\tfrac{1}{3}gh}.\quad \textit{Ans.}$$

11. The effect is the same if the point is kept fixed and the ball is projected horizontally in the opposite direction. At the highest point,

$$T + mg = \frac{mv^2}{l}$$

where v is the speed at the highest point.

By 'energy conservation' at the highest point

$$\tfrac{1}{2}mV^2 = mg(2l) + \tfrac{1}{2}mv^2$$

where V is the required velocity

or $$V^2 = 4gl + v^2$$

$$\therefore \quad T + mg = \frac{m(V^2 - 4gl)}{l}.$$

For just going in a vertical circle the string must not slacken at the highest point, that is, $T \geqslant 0$

or $$\frac{m(V^2 - 4gl)}{l} \geqslant mg, \quad \text{or} \quad V^2 \geqslant 5gl, \quad \text{or} \quad V \geqslant \sqrt{5gl}.$$

$$\therefore \quad V_{min} = \sqrt{5gl}. \quad \textit{Ans.}$$

Let T be the tension in the horizontal position.

Then $$T = \frac{mv^2}{l}.$$

By 'energy conservation' in this position

$$\tfrac{1}{2}mv^2 + mgl = \tfrac{1}{2}mV^2_{min}$$

or $$v^2 + 2gl = 5gl, \quad \text{or} \quad v^2 = 3gl.$$

$$\therefore \quad T = m.\frac{3gl}{l} = 3mg. \quad \textit{Ans.}$$

12. By conservation of energy

$$m_1g(r_1 - r_1\cos\alpha) + m_2g(r_2 - r_2\cos\alpha) = \tfrac{1}{2}m_1\omega^2r_1^2 + \tfrac{1}{2}m_2\omega^2r_2^2$$

$(\because \quad v_1 = \omega r_1$ and $v_2 = \omega r_2)$

or $$2g(1 - \cos\alpha)(m_1r_1 + m_2r_2) = (m_1r_1^2 + m_2r_2^2)\omega^2$$

or $$4g\sin^2\frac{\alpha}{2}(m_1r_1 + m_2r_2) = (m_1r_1^2 + m_2r_2^2)\omega^2$$

or $$\omega = 2\sin\frac{\alpha}{2}\sqrt{g \times \frac{m_1r_1 + m_2r_2}{m_1r_1^2 + m_2r_2^2}}.$$

$$\therefore \quad v_1 = 2r_1\sin\frac{\alpha}{2}\sqrt{g \times \frac{m_1r_1 + m_2r_2}{m_1r_1^2 + m_2r_2^2}}. \quad \textit{Ans.}$$

$$v_2 = 2r_2\sin\frac{\alpha}{2}\sqrt{g \times \frac{m_1r_1 + m_2r_2}{m_1r_1^2 + m_2r_2^2}}. \quad \textit{Ans.}$$

13. The forces acting on the bead : (i) mg, downward, (ii) N, normal reaction and (iii) μN up the rod.

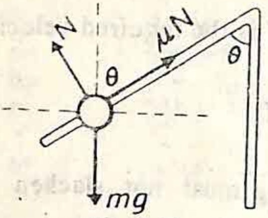


Fig. 5.58

The radius of the circle in which m moves $= l\sin\theta$

centripetal force $= \mu N\sin\theta - N\cos\theta$

$$= m\omega^2 l\sin\theta$$

or $N(\mu\sin\theta - \cos\theta) = m\omega^2 l\sin\theta$ .. (i).

Since there is no vertical acceleration

$$mg = \mu N\cos\theta + N\sin\theta$$

or $mg = N(\mu\cos\theta + \sin\theta)$ .. (ii).

From (i) and (ii), eliminating N we have

$$\frac{\mu\sin\theta - \cos\theta}{\mu\cos\theta + \sin\theta} = \frac{\omega^2 l\sin\theta}{g}$$

or $$l = \frac{g}{\omega^2} \cdot \frac{\mu\sin\theta - \cos\theta}{\sin\theta(\mu\cos\theta + \sin\theta)}. \quad Ans.$$

14. Let us consider the first vertical upward position of string when its tension is T.

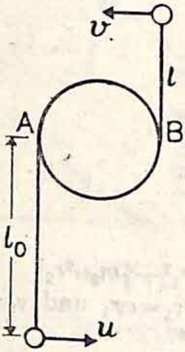


Fig. 5.59

By 'conservation of energy'

$$\tfrac{1}{2}mu^2 - mgl_0 = \tfrac{1}{2}mv^2 + mgl$$

or $u^2 - 2gl_0 = v^2 + 2gl$.

The relation of l and l_0 is

$$l = l_0 - \pi r.$$

$$\therefore \quad u^2 = 2gl_0 + v^2 + 2g(l_0 - \pi r)$$

$$= 4gl_0 - 2\pi rg + v^2.$$

From 'dynamics of circular motion' (instantaneous centre of rotation is B)

$$T + mg = \frac{mv^2}{l}$$

or $$T = \frac{m}{l}(u^2 - 4gl_0 + 2\pi rg) - mg.$$

If the tension in the first vertical position is zero, it will not be zero in the subsequent motion because tension will increase with increase of speed. Therefore, for not slackening in any position we have for minimum value of u,

or $$T = 0$$

or $$0 = u^2 - 4gl_0 + 2\pi rg - gl$$

$$u^2 = 4gl_0 - 2\pi rg + g(l_0 - \pi r)$$

or $$u = \sqrt{5gl_0 - 3\pi rg} = \sqrt{5g \times \cdot 28 - 3\pi \times \cdot 04 \times g}$$

$$= \sqrt{13 \cdot 72 - 3 \cdot 7}$$

$$= 3 \cdot 16 \text{ m/s.} \quad Ans.$$

15. The forces acting on m : (i) mg downward,

(ii) T tension,

(iii) N, normal reaction.

Resolving along horizontal and vertical

$$T\sin\theta=\frac{mv^2}{l\sin\theta}, \quad (\because \text{ radius of the circle in which } m \text{ moves } = l\sin\theta)$$

$$R+T\cos\theta=mg \quad (\because \text{ there is no vertical acceleration}).$$

The condition for the ball to remain on the table is

$$R\geqslant 0$$

or

$$mg\geqslant T\cos\theta$$

or

$$mg\geqslant\frac{mv^2\cos\theta}{l\sin^2\theta}, \quad \text{or} \quad v^2\leqslant\frac{gl\sin^2\theta}{\cos\theta}.$$

Now

$$\sin\theta=\frac{\sqrt{i^2-h^2}}{l} \quad \text{and} \quad \cos\theta=\frac{h}{l}.$$

$$\therefore \quad v^2\leqslant\frac{gl\frac{l^2-h^2}{l^2}}{h/l}$$

$$v^2\leqslant\frac{g}{h}(l^2-h^2)$$

$$v_{max}=\sqrt{\frac{g}{h}(l^2-h^2)} \quad . \quad \textit{Ans.}$$

$$T=\frac{mv^2{}_{max}}{l\sin^2\theta}=\frac{\frac{mg}{h}(l^2-h^2)}{l.\frac{l^2-h^2}{l^2}}=\frac{mgl}{h}. \quad \textit{Ans.}$$

16. For A, we have $\quad T=m\omega_1{}^2l_1.$

For B, $\quad T\sin\theta=m\omega_2{}^2l_2\sin\theta$

or $\quad T=m\omega_2{}^2l_2$

and $\quad T\cos\theta=mg, \quad (\because \text{ there is no vertical acceleration of } B)$

or

$$\cos\theta=\frac{mg}{T}.$$

$$\therefore \quad m\omega_2{}^2l_2=T=m\omega_1{}^2l_1$$

or $\quad \omega_2{}^2l_2=\omega_1{}^2l_1, \quad$ or $\quad l_1:l_2=\omega_2{}^2:\omega_1{}^2.$ *Proved.*

Motion is possible so long $\quad T>mg$

$(\because \cos\theta$ cannot be greater than 1)

or

$$m\omega_1{}^2l_1>mg$$

or $$\omega_1^2 \cdot \frac{\omega_2^2 l}{\omega_1^2+\omega_2^2} > g,$$

$$\left(\because \ \frac{l_1}{l_2} = \frac{\omega_2^2}{\omega_1^2}, \ \therefore \ \frac{l_1}{l_1+l_2} = \frac{\omega_2^2}{\omega_1^2+\omega_2^2} \right)$$

or $\dfrac{\omega_1^2+\omega_2^2}{\omega_1^2\omega_2^2} < \dfrac{l}{g}$, or $\dfrac{1}{\omega_1^2} + \dfrac{1}{\omega_2^2} < \dfrac{l}{g}$. *Proved.*

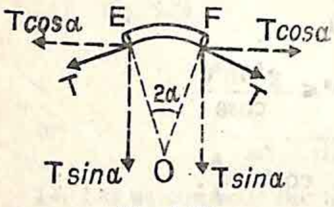


Fig. 5.60

17. Consider an element subtending an angle 2α at the centre. Let T be the tension of the tube.

Mass of water in the element $= (\pi r^2 \times 2\alpha R) \times \rho$

where ρ = density of water.

Centripetal force on the moving water $= 2T\sin\alpha = 2T\alpha$ (since $\sin\alpha = \alpha$)

$$\therefore \ 2T\alpha = (\pi r^2 \times 2\alpha R\rho)\frac{v^2}{R}$$

or $$T = \pi r^2 \rho v^2. \quad \textit{Ans.}$$

18. From dynamics of circular motion

$$T - mg\cos\theta = \frac{mv^2}{l}, \quad \text{or} \quad T = mg\cos\theta + \frac{mv^2}{l}.$$

From conservation of energy,

$$\tfrac{1}{2}mv^2 + mg(l - l\cos\theta) = \tfrac{1}{2}mv_0^2$$

where v_0 is the speed at the lowest position.

$$\therefore \ v_0^2 = 2gl(1-\cos\theta).$$

$$\therefore \ T = {\cdot}1 \times 9{\cdot}8\cos 60^\circ + \frac{{\cdot}1 \times 2^2}{1} = {\cdot}98 \times {\cdot}5 + {\cdot}4 = {\cdot}89 \text{ N}. \quad \textit{Ans.}$$

$$v_0 = \sqrt{2 \times 9{\cdot}8 \times 1(1-\cos 60^\circ)} = 3{\cdot}1 \text{ m/s}. \quad \textit{Ans.}$$

19. From 'conservation of energy'

$$mg(l - l\cos\theta_0) = \tfrac{1}{2}mv^2 + mg(l - l\cos\theta)$$

or $$2gl(\cos\theta - \cos\theta_0) = v^2. \quad \ldots \text{(i)}$$

From 'dynamics of circular motion'

$$T - mg\cos\theta = \frac{mv^2}{l}$$

or $$T - mg\cos\theta = \frac{m}{l}.2gl(\cos\theta - \cos\theta_0)$$

or $$T = 3mg\cos\theta - 2mg\cos\theta_0 = mg(3\cos\theta - 2\cos\theta_0). \quad \textit{Ans.}$$

Obviously tension is maximum when $\theta = 0°$.

$$\therefore \quad T_{max} = mg(3 - 2\cos\theta_0).$$

Here $T_{max} = 80g$ and $m = 40$ kg

$$\therefore \quad 80g = 40g(3 - 2\cos\theta_0)$$

or $\cos\theta_0 = \frac{1}{2}$, or $\theta_0 = 60°$. *Ans.*

Simple Harmonic Motion

1. According to the law at $t = 0$, $x = 0$, that is, the particle is at the mean position. If t is exactly divisible by $\frac{T}{4}$, that is, $t = n\frac{T}{4}$ where n is an integer, the distance covered is na because in every quarter cycle the particle describes a distance 'a'. If t is not exactly divisible by $\frac{T}{4}$, then we can write $t = n\frac{T}{4} + t'$ where t' is additional time less than $\frac{T}{4}$.

The distance covered will be na plus the distance described in t'.

Thus
$$s = na + a\sin\omega\left(t - \frac{nT}{4}\right)$$
$$= na + a\sin\left(\omega t - \frac{n\pi}{2}\right)$$

where n is the number of quarter cycles in t. *Ans.*

We have $x = 3\sin\omega t + 4\cos\omega t$.

Put $3 = a\cos\delta$ and $4 = a\sin\delta$.

Squaring and adding, $a = 5$.

Dividing, $\tan\delta = \frac{4}{3}$.

$$\therefore \quad x = a(\sin\omega t\cos\delta + \cos\omega t\sin\delta)$$

or
$$x = a\sin(\omega t + \delta) = 5\sin(\omega t + \tan^{-1}\tfrac{4}{3})$$
$$= 5\sin(\omega t + 53°).$$

Thus, amplitude $= 5$, frequency $= \frac{\omega}{2\pi}$ and epoch $= 53°$. *Ans.*

2. Here $\pi = \omega$, or $T = 2$ s; $\therefore$ quarter cycle $= \frac{T}{4} = \cdot 5$ s.

Now $\frac{2\cdot 9}{\cdot 5} = 5 + \cdot 4$. That is, there are 5 quarter cycles and $\cdot 4$ s additional time.

$$\therefore \quad \text{Distance described} = 10\times 5 + 10\sin\pi\times {\cdot}4$$
$$= 50 + 10\times\sin 72^\circ$$
$$= 50 + 10\times{\cdot}95 = 59{\cdot}5 \text{ cm.} \quad \textit{Ans.}$$

3. Let v be the volume of the hydrometer submerged.

Then $\quad mg = v\rho g \quad$ by law of floatation

or $\quad m = v\rho. \qquad \ldots$ (i)

Let the hydrometer be depressed a little and then released. Let x be the additional length of the hydrometer stem submerged at the instant t.

Unbalanced upward force on the hydrometer

$$= \text{upthrust of water} - \text{gravitational pull}$$
$$= (v + \pi r^2 x\ \rho g - mg$$
$$= v\rho g + \pi r^2 x\rho g - v\rho g$$
$$= \pi r^2 x\rho g.$$

$$\text{Force} = \text{mass}\times\text{acceleration}$$

$$\therefore \quad -\pi r^2 x\rho g = m\times\frac{d^2x}{dt^2} \quad (\because \ -\pi r^2 x\rho g \text{ is the downward force})$$

or
$$\frac{d^2x}{dt^2} = -\frac{\pi r^2\rho g}{m}x.$$

$$\therefore \quad \omega = \frac{\pi r^2\rho g}{m}.$$

$$T = 2\pi\sqrt{\frac{m}{\pi r^2\rho g}}$$

$$= 2\pi\sqrt{\frac{50\times 10^{-3}}{\pi\times 3{\cdot}2^2\times 10^{-6}\times 1000\times 9{\cdot}8}} = 2{\cdot}5 \text{ s.} \quad \textit{Ans.}$$

4. Maximum acceleration $= \omega^2 a$.

When the limiting frictional force equals the maximum force, the body begins to slide.

$$\therefore \quad \omega^2 a = \mu g, \quad \text{or} \quad \mu = \omega^2\frac{a}{g}$$

$$= \left(\frac{2\pi}{T}\right)^2\times\frac{a}{g}$$

$$= \left(\frac{2\pi}{1}\right)^2\times\frac{{\cdot}1}{9{\cdot}8} = {\cdot}4. \quad \textit{Ans.}$$

5. There are three forces on B : (i) $m_2 g$ downward,
(ii) N, normal reaction upward,
(iii) f, force of the spring.

The force f is fluctuating both in magnitude and direction. When A is at its lowest position, A is pressed against the surface by the spring and when A is at its highest position, A experiences upward force. If f_0 is the amplitude of the force, then

$$m_1g+m_2g=N\pm f_0$$

$$N=m_1g+m_2g\mp f_0.$$

Now $$f_0=m_1a\omega^2$$

$$N=m_1g+m_2g\pm m_1a\omega^2$$

or $$N=1g+4g\pm 1\times 1{\cdot}6\times 10^{-2}\times 25^2$$

or $$N=49\pm 10=59\text{ N} \quad\text{or}\quad 39\text{ N}. \quad \textit{Ans.}$$

6. In the equilibrium position normal reactions of the wheels are the same. Let each of them be N.

Then $$2N=mg.$$

Let the plank be displaced to the right and then released. The reactions of the wheels are no more equal. Let them be N_1 and N_2.

The corresponding frictional forces are kN_1 and kN_2 to the right and left respectively.

Unbalanced force to the right $=kN_1-kN_2$
$=k(N_1-N_2)$.

If x is the displacement to the right then

$$k(N_1-N_2)=m\frac{d^2x}{dt^2}\,. \qquad \text{(i)}$$

As there is no vertical acceleration

$$mg=N_1+N_2.$$

Considering rotational equilibrium of the plank about A

$$N_2l=mg\left(\frac{l}{2}+x\right)$$

or $$N_2=\tfrac{1}{2}mg+\frac{mxg}{l}$$

and $$N_1=\tfrac{1}{2}mg-\frac{mxg}{l}$$

$$\therefore\quad N_2-N_1=\frac{2mg}{l}$$

$$\therefore\quad m\frac{d^2x}{dt^2}=-k\frac{2xmg}{l}.$$

$\therefore\ \dfrac{d^2x}{dt^2}\propto -x$ and the motion is simple harmonic.

Here $$\omega^2=\frac{2kg}{l}.$$

$$T=2\pi\sqrt{\frac{l}{2kg}}$$

or $$T=\pi\sqrt{\frac{2l}{kg}}.\quad \textit{Ans.}$$

7. Let $2l$ be the length of the spring m in the equilibrium. Let the mass be displaced by x to the right. Then leftward force

$$=kx+kx=2kx.$$

$\therefore$ Force to the right $=-2kx$.

There are two accelerations of m : one is the centripetal acceleration $=\omega^2x$ and the other is $\frac{d^2x}{dt^2}$.

$$\therefore \quad -2kx=m\left(\frac{d^2x}{dt^2}-\omega^2x\right)$$

or $$m\frac{d^2x}{dt^2}=-2kx+m\omega^2x$$

or $$\frac{d^2x}{dt^2}=-\left(\frac{2k}{m}-\omega^2\right)x.$$

Thus, $\frac{d^2x}{dt^2}\propto -x$ and the motion is simple harmonic.

$$\therefore \quad \left(\frac{2\pi}{T}\right)^2=\frac{2k}{m}-\omega^2$$

or $$T=\frac{2\pi}{\sqrt{\frac{2k}{m}-\omega^2}}.\quad \textit{Ans.}$$

T is imaginary when $\frac{2k}{m}<\omega^2$. Thus at frequencies $\omega\geqslant\sqrt{\frac{2k}{m}}$, there will be no oscillations of the sleeve.

8. Let T be the tension of each thread. Resolving the tensions along the vertical and horizontal we have

$$T\cos\alpha+T\cos\alpha=mg$$

(as there is no vertical acceleration)

or $$2T\cos\alpha=mg \quad \text{or} \quad T=\tfrac{1}{2}mg$$

($\because$ $\cos\alpha=1$ when α small).

Restoring torque on the rod $= T\sin\alpha \times l'$ where l' is the length of the rod. This is in the negative direction, that is, decreasing direction of θ.

$\therefore$ Torque (τ) in the +ve direction $= -T\sin\alpha l'$.

$\because$ Torque = moment of inertia × angular acceleration,

$$-T\sin\alpha l' = (\tfrac{1}{12}ml'^2)\times\frac{d^2\theta}{dt^2}$$

or $$\frac{d^2\theta}{dt^2} = -\frac{12T\alpha}{ml'} \qquad (\because \ \sin\alpha = \alpha).$$

From the figure $\qquad AA' = l\alpha$

and also $\qquad AA' = \theta\times\frac{l'}{2}$

$$\therefore \quad l\alpha = \theta.\frac{l'}{2} \quad \text{or} \quad \alpha = \frac{l'}{2l}.\theta$$

$$\therefore \quad \frac{d^2\theta}{dt^2} = -\frac{12\times\frac{1}{2}mg}{ml'}\times\frac{l'}{2l}.\theta \qquad (\because \ T = \tfrac{1}{2}mg)$$

$$\therefore \quad \frac{d^2\theta}{dt^2} = -\frac{3g}{l}\theta.$$

Thus, $\frac{d^2\theta}{dt^2} \propto -\theta$ and motion is simple harmonic.

Here, $$\left(\frac{2\pi}{T}\right)^2 = \frac{3g}{l}$$

or $$T = 2\pi\sqrt{\frac{l}{3g}}. \quad \textit{Ans.}$$

9. Obviously the centre of mass of the ring moves in a circle of radius $(R-r)$. The forces acting on the ring are : (i) mg, downward,
(ii) N, normal reaction,
(iii) F, force of friction.

Centripetal force $= N - mg\cos\theta$
$= m\omega^2(R-r)$.

Tangential force $= F - mg\sin\theta$
$= ma_\tau$ where a_τ = tangential acceleration

$$= \frac{dv}{dt} = \frac{d}{dt}\omega(R-r) = (R-r)\frac{d\omega}{dt}.$$

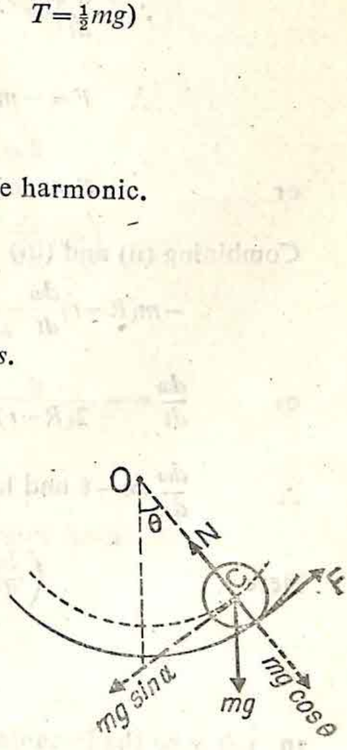


Fig. 5.61

$$\therefore \quad N - mg\cos\theta = m\omega^2(R-r). \qquad \text{.. (i)}$$

and
$$F - mg\sin\theta = m(R-r)\frac{d\omega}{dt}. \qquad \text{.. (ii)}$$

Let the ring rotate through φ about its axis when position vector of its centre (that is, the line OC) describes angle θ at O.

Considering rotational motion about its axis we have

$$\tau = -Fr$$

$$\therefore \quad -Fr = (mr^2)\frac{d^2\varphi}{dt^2} \qquad \text{(torque = m i.} \times \text{ang. acc.)}$$

or
$$F = -mr\frac{d^2\varphi}{dt^2}.$$

Since the ring rolls without slipping

$$r(\theta+\varphi) = R\theta \quad \text{or} \quad \varphi = \frac{R-r}{r}\theta$$

$$\therefore \quad \frac{d^2\varphi}{dt^2} = \frac{R-r}{r}\cdot\frac{d^2\theta}{dt^2} = \frac{R-r}{r}\cdot\frac{d\omega}{dt} \quad \left(\because \quad \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt}\right)$$

$$\therefore \quad F = -mr \times \frac{R-r}{r}\,\frac{d\omega}{dt}$$

or
$$F = -m(R-r)\frac{d\omega}{dt}. \qquad \text{.. (iii)}$$

Combining (ii) and (iii)

$$-m(R-r)\frac{d\omega}{dt} - mg\sin\theta = m(R-r)\frac{d\omega}{dt}$$

or
$$\frac{d\omega}{dt} = -\frac{g}{2(R-r)}\sin\theta = -\frac{g}{2(R-r)} \times \theta.$$

$$\therefore \quad \frac{d\omega}{dt} \propto -\theta \text{ and hence motion is simple harmonic.}$$

Here
$$\left(\frac{2\pi}{T}\right)^2 = \frac{g}{2(R-r)}$$

or
$$T = 2\pi\sqrt{\frac{2(R-r)}{g}}. \quad \textit{Ans.}$$

10. Let x be the instantaneous displacement (downward) of the suspended mass M. Then energy of the system is given by

$$\tfrac{1}{2}Mv^2 + \tfrac{1}{2}kx^2 + \text{k.e. of the spring} = \text{a constant.}$$

To calculate energy of the spring consider an element of it at a distance z from the upper end and thickness dz.

Mass of the element $=\left(\frac{m}{l}dz\right)$ where l is the length of the spring.

Displacement of the element $=\frac{x}{l}.z$.

$\therefore$ Velocity of the element $=\frac{z}{l}.\frac{dx}{dt}=\frac{z}{l}v \quad \left(\because \frac{dx}{dt}=v\right)$;

$\therefore$ k.e. of the element $=\frac{1}{2}\left(\frac{m}{l}dz\right).\left(\frac{z}{l}v\right)^2$

$$=\frac{1}{2}\frac{mv^2}{l^3}z^2dz;$$

$\therefore$ k.e. of the spring $=\frac{1}{2}\frac{mv^2}{l^3}\int_0^l z^2dz=\frac{1}{6}mv^2$

$\therefore \quad \frac{1}{2}Mv^2+\frac{1}{2}kx^2+\frac{1}{6}mv^2=$ a constant.

Differentiating, $Mv\frac{dv}{dt}+kx\frac{dx}{dt}+\frac{1}{3}mv\frac{dv}{dt}=0$

or $\quad Mv\frac{dv}{dt}+kxv+\frac{1}{3}mv.\frac{dv}{dt}=0 \quad \left(\because \frac{dx}{dt}=v\right)$

or
$$\left(M+\frac{m}{3}\right)\frac{dv}{dt}=-kx$$

or
$$\frac{dv}{dt}=-\frac{k}{M+\frac{m}{3}}x.$$

Thus $\frac{dv}{dt}\propto -x$ and hence motion is simple harmonic.

Here
$$\left(\frac{2\pi}{T}\right)^2=\frac{k}{M+\frac{m}{3}}$$

or
$$T=2\pi\sqrt{\frac{M+\frac{m}{3}}{k}}.$$ *Proved.*

Consider a small angular displacement by θ in the clockwise direction.

Then unbalanced torque (τ) in the clockwise direction

$$= m'gR - mgR\sin(\alpha+\theta).$$

Moment of inertia of the system $= \frac{1}{2}MR^2 + m'R^2 + mR^2$

$$\therefore \quad m'gR - mgR\sin(\alpha+\theta)$$

$$= (\tfrac{1}{2}MR^2 + m'R^2 + mR^2)\frac{d^2\theta}{dt^2}.$$

Putting $\quad m' = m\sin\alpha$ we have

$$mg\sin\alpha R - mgR\sin(\alpha+\theta)$$

$$= (\tfrac{1}{2}MR^2 + m\sin\alpha R^2 + mR^2)\frac{d^2\theta}{dt^2}$$

or

$$mgR\sin\alpha - mgR(\sin\alpha\cos\theta + \cos\alpha\sin\theta)$$

$$= \tfrac{1}{2}R^2(M + 2m\sin\alpha + 2m)\frac{d^2\theta}{dt^2}$$

or

$$mgR\sin\alpha - mgR\sin\alpha - mgR\cos\alpha\theta$$

$$= \tfrac{1}{2}R^2(M + 2m\sin\alpha + 2m)\frac{d^2\theta}{dt^2}$$

($\because \; \sin\theta = \theta$ and $\cos\theta = 1$ when θ is small)

or

$$-2mg\cos\alpha\theta = [MR + 2mR(1+\sin\alpha)]\frac{d^2\theta}{dt^2}.$$

Thus $\dfrac{d^2\theta}{dt^2} \propto -\theta$ and hence the motion is simple harmonic.

Here

$$\left(\frac{2\pi}{T}\right)^2 = \frac{2mg\cos\alpha}{MR + 2mR(1+\sin\alpha)}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{2mg\cos\alpha}{MR + 2mR(1+\sin\alpha)}}. \quad \textit{Ans.}$$

13. The electric field at a point on the axis of a charged ring is given by

$$E = \frac{1}{4\pi\epsilon_0} \cdot \frac{Qx}{(a^2+x^2)^{\frac{3}{2}}}$$

(see deduction of this formula in electrostatics).

At a point close to the centre

$$E = \frac{1}{4\pi\epsilon_0} \cdot \frac{Qx}{a^3}. \qquad (\because \; a \gg x)$$

Let us consider a small displacement of charge q from the centre by x.

11. Consider an instantaneous displacement of the centre of the cylinder by x to the right and let the cylinder rotate through θ. The forces acting on it are : (i) Mg, downward, (ii) kx, spring force to the left, (iii) N, normal reaction, (iv) F, force of friction to the right.

Considering 'linear motion of the centre of mass' of the cylinder

$$-kx+F=M\frac{d^2x}{dt^2}. \qquad \text{.. (i)}$$

Considering rotational motion we have

$$\tau(\text{torque})=-F.r.$$

$$\therefore \quad -Fr=(\tfrac{1}{2}Mr^2)\frac{d^2\theta}{dt^2} \qquad (\text{torque}=\text{m.i.}\times\text{ang. acc.})$$

or

$$F=-\tfrac{1}{2}Mr\frac{d^2\theta}{dt^2}.$$

Since it rolls without slipping

$$x=r\theta$$

or

$$\frac{d^2x}{dt^2}=r\frac{d^2\theta}{dt^2}.$$

$$\therefore \quad F=-\tfrac{2}{5}M\frac{d^2x}{dt^2}. \qquad \text{.. (ii)}$$

Combining (i) and (ii)

$$-kx-\tfrac{1}{2}M\frac{d^2x}{dt^2}=M\frac{d^2x}{dt^2}$$

or

$$-kx=\tfrac{3}{2}M\frac{d^2x}{dt^2}$$

or

$$\frac{d^2x}{dt^2}=-\frac{2k}{3M}x.$$

Thus $\frac{d^2x}{dt^2} \propto -x$ and hence motion is simple harmonic.

Here,

$$\left(\frac{2\pi}{T}\right)^2=\frac{2k}{3M}$$

$$T=2\pi\sqrt{\frac{3M}{2k}}. \quad \textit{Proved.}$$

12. Considering 'rotational equilibrium about O' we have

$$m'gR=mgR\sin\alpha \text{ where } m'=\text{mass of } A$$

or

$$m'=m\sin\alpha.$$

Then force on $q=\frac{1}{4\pi\epsilon_0}\cdot\frac{Qqx}{a^3}$ in the negative direction of x

$$=-\frac{1}{4\pi\epsilon_0}\cdot\frac{Qqx}{a^3} \text{ in the positive direction}$$

$$\therefore\quad -\frac{1}{4\pi\epsilon_0}\cdot\frac{Qq}{a^3}x=m\frac{d^2x}{dt^2}$$

or

$$\frac{d^2x}{dt^2}=-\frac{1}{4\pi\epsilon_0}\cdot\frac{Qq}{a^2m}x$$

$\therefore\quad \frac{d^2x}{dt^2}\propto -x$ and hence motion is simple harmonic.

Here

$$\left(\frac{2\pi}{T}\right)^2=\frac{1}{4\pi\epsilon_0}\cdot\frac{Qq}{a^3m}.$$

$$\therefore\ f=\frac{1}{2\pi}\sqrt{\frac{Qq}{4\pi\epsilon_0 a^3m}}.\ \textit{Ans.}$$

14. The frequency of oscillation of 'spring controlled pendulum' depends only on the mass suspended $\left(T=2\pi\sqrt{\frac{m}{k}}\right)$.

$$\therefore\quad T=2\pi\sqrt{\frac{m_2}{k}}$$

ω (angular frequency)$=\frac{2\pi}{T}=\sqrt{\frac{k}{m_2}}$. *Ans.*

Let l_0 be the natural length of the spring. In the equilibrium position (that is, when there is no vibration) its length is

$$\left(l_0+\frac{m_2g}{k}\right).$$

With (m_1+m_2), its length was $\left(l_0+\frac{m_1+m_2}{k}g\right)$,

$$\therefore\ \text{amplitude of oscillation}=\left(l_0+\frac{m_1+m_2}{k}g\right)-\left(l_0+\frac{m_2g}{k}\right)$$

$$=\frac{m_1}{k}g.\ \textit{Ans.}$$

15. The forces acting on the block are : (i) mg, downward, (ii) N, normal reaction.

τ (torque on the block)

$= mgR\sin\theta$ in $-$ve direction of θ

$= -mgR\sin\theta$ in $+$ve direction of θ.

$\because$ torque = m.i. × ang. acc.,

$$-mgR\sin\theta = mR^2 \times \frac{d^2\theta}{dt^2}$$

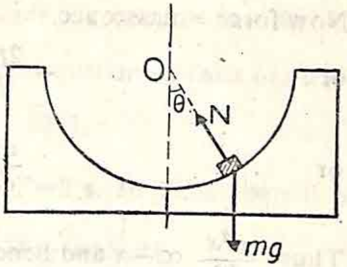


Fig. 5.62

or
$$\frac{d^2\theta}{dt^2} = -\frac{g}{R}\theta.$$

Thus $\frac{d^2\theta}{dt^2} \propto -\theta$ and hence the motion is simple harmonic.

Here
$$\left(\frac{2\pi}{T}\right)^2 = \frac{g}{R}$$

or
$$T = 2\pi\sqrt{\frac{R}{g}}. \quad Ans.$$

16. Since the system is free from external force along the horizontal, the acceleration of the centre of mass of the system is zero.

$$a_{cm} = \frac{am + a'M}{m+M}, \text{ where } a = \text{acceleration of } m$$
$$a' = \text{acceleration of } M$$

or $am = -a'M$.

The acceleration of m relative to $M = a - (-a')$

$$= a + a' = a + \frac{am}{M}$$

$$= a\left(1 + \frac{m}{M}\right).$$

17. Let x be the small downward displacement, unbalanced force on $m = f\sin\theta + f\sin\theta$ in the negative direction of x

$= -2f\sin\theta$ in the $+$ve direction

$= -2f\theta$ ($\because$ θ small)

$= -2f\frac{x}{l}$.

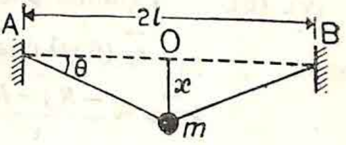


Fig. 5 63

Now force = mass × acc.

or
$$-\frac{2fx}{l}=m\times\frac{d^2x}{dt^2}$$

or
$$\frac{d^2x}{dt^2}=-\frac{2f}{lm}x.$$

Thus $\frac{d^2x}{dt^2}\propto -x$ and hence the motion is simple harmonic.

Here
$$\left(\frac{2\pi}{T}\right)^2=\frac{2f}{lm}.$$

or
$$T=2\pi\sqrt{\frac{ml}{2f}}.\quad \textit{Ans.}$$

Solutions to Test Questions

1. (i). (c). Here a_τ (tangential acceleration) $= 3\ \text{m/s}^2$

and a_r (centripetal acceleration) $=\frac{30^2}{225}=4\ \text{m/s}^2$

$$\therefore\quad a=\sqrt{a_\tau{}^2+a_r{}^2}=\sqrt{3^2+4^2}=5\ \text{m/s}^2.$$

(ii). (c).

(iii). (a) Centripetal force = frictional force; $\therefore\ f_r=\frac{mv^2}{r}$.

But $f_r\leqslant f_{lim}$ where f_{lim} = limiting frictional force $=\mu\, mg$

or
$$\frac{mv^2}{r}\leqslant\mu\, mg$$

or $v^2\leqslant\mu\, gr.\quad\therefore\quad v_{max}=\sqrt{\mu gr}=\sqrt{\cdot25\times10\times1000}$
$=50$ m/s.

(iv) (b), because tendency to overturn is proportional to the square of speed.

(v). (b). $\vec{r}_i$ (initial position vector) $=R\,\vec{i}$

$\vec{r}_f$ (final position vector) $=-R\,\vec{j}$

$$\therefore\quad \langle\vec{v}\rangle=\frac{-R\vec{j}-R\vec{i}}{\tau}.\quad\therefore\quad |\langle\vec{v}\rangle|=\frac{\sqrt{2}R}{\tau}.$$

(vi). (d). $v=\omega\sqrt{a^2-x^2}=\frac{2\pi}{T}\sqrt{a^2-x^2}=\frac{2\pi}{3}\sqrt{5^2-4^2}=6\ \text{m/s}.$

(vii). (a). $T=2n\sqrt{\frac{m}{k}}$. The force constant is inversely proportional to the length. Hence force constant of each piece is $4k$.

$$\therefore \quad T'=2\pi\sqrt{\frac{m}{4k}}. \quad \therefore \quad T'=T/2.$$

(viii). (b). $\therefore$ Here $\omega=\pi$; $\therefore$ $T=2$ s. In 2·5 s there is only one quarter cycle, and one full cycle.

$$\therefore \quad \text{Distance}=4a+(a-a\cos\pi t)$$
$$=4a+a\left(1-\cos\frac{\pi}{2}\right)=5a.$$

(ix). (c). $\mu mg=m\omega^2 r$ or $\omega^2 r=$ a constant

$$\therefore \quad \omega^3\times 3=(2\omega)^2\times r$$

or $\quad r=\cdot 75$ cm.

(x). (b). $\frac{f_x}{f_y}=\frac{n_y}{n_x}$. $\therefore$ $f_y=\frac{n_x}{n_y}\times f_x=\frac{3}{5}\times 400=240.$

2. Suppose d is the distance of the obstacle from the car. To give a circular motion a centripetal force of magnitude $\frac{mv^2}{d}$ is needed. To stop it within d directly, force needed is

$$ma=m\frac{v^2}{2d}=\tfrac{1}{2}\frac{mv^2}{d} \quad (\because \; v^2=2ad).$$

So it is easier to stop it directly by applying brake.

(ii) Because unless a force acts on the body directed inwards, a body will move in a straight line according to Newton's first law of motion.

(iii) Because in the beginning the earth was a body of molten materials. Due to centrifugal force the molten mass was thrown away from the axis and so it was flattened along its axis.

(iv) False. The general motion is elliptical. A circle is a particular case of ellipse when its major axis is equal to its minor axis. In general, motion is described by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2xy}{ab}\cos\delta=\sin^2\delta$$

(for deduction of this see 'Introductory Physics' Part I, simple harmonic motion chapter).

When $a=b$ and $\delta=\pi/2$,

$$x^2+y^2=a^2.$$

(v) We can account for the centripetal force, that is, we can say what body in the environment exerts this force and so it is real. But we cannot account for the centrifugal force and hence it is a 'pseudo force'.

3. We have $5=k\times{\cdot}1$ or $k=50$ N/m.

(∵ Force $=kx$ in a spring)

We have for a 'spring operated pendulum'

$$T=2\pi\sqrt{\frac{m}{k}}.\quad\therefore\quad \tfrac{1}{2}=2\pi\sqrt{\frac{m}{50}}$$

or $$m=\frac{50}{16\pi^2}={\cdot}31\text{ kg}$$

$$=310\text{ gm.}\quad \textit{Ans.}$$

$$V_{max}=a\omega=a.\frac{2\pi}{T}={\cdot}01\times\frac{2\pi}{\frac{1}{2}}={\cdot}01\times4\pi={\cdot}125\text{ m/s}=12{\cdot}5\text{ cm/s.}$$

4. According to the formula, the position of the particle is at the centre of the path. It goes by 7 units to the right and then by another 7 units back to the initial position and then it goes to the negative side by $\frac{7}{2}$ units for the first time.

So the minimum time is $2\times\frac{T}{4}$ plus the additional time (t) to cover

$\frac{7}{2}$ units on the negative side.

Here $\omega=\frac{\pi}{2}$ or $\frac{2\pi}{T}=\frac{\pi}{2}$ or $T=4$ s.

$$\therefore\quad t_{min}=2\times\tfrac{4}{4}+t=2+t.$$

Now $$\frac{7}{2}=7\sin\frac{\pi}{2}t$$

or $$\frac{\pi}{2}t=\sin30^\circ=\sin\frac{\pi}{6}$$

$$\therefore\frac{\pi}{2}t=\frac{\pi}{6}\quad\text{or}\quad t=\frac{1}{3}\text{ s}$$

$$\therefore\quad t_{min}=2+\tfrac{1}{3}=2{\cdot}33\text{ s.}\quad\textit{Ans.}$$

5. Let l be the length of each column. Let ρ be the density of the liquid and A is the area of cross-section of the tube. Then mass of the liquid '$2lA\rho$'.

Let one column be depressed by x. Then the other column will go up by x. So the level difference is $2x$. This corresponds to pressure difference '$2x\rho g$'.

$\therefore$ Force on the liquid mass

$$= 2x\rho gA \text{ in the negative direction of } x$$
$$= -2x\rho gA \text{ in the positive direction of } x.$$

$$\therefore \quad -2x\rho gA = (2lA\rho)\frac{d^2x}{dt^2}$$

or
$$\frac{d^2x}{dt^2} = -\frac{g}{l}x.$$

Thus $\frac{d^2x}{dt^2} \propto -x$ and hence the motion is simple harmonic.

Here
$$\left(\frac{2\pi}{T}\right)^2 = \frac{g}{l}$$

or
$$T = 2\pi\sqrt{\frac{l}{g}}. \quad Ans.$$

6. This is a problem on 'conical pendulum'.

Let θ be the angle made by the string with the vertical.

Then $\cos\theta = \frac{3{\cdot}2}{4} = {\cdot}8$ or $\theta = 37°$.

Radius of the circle of motion $= \sqrt{4^2 - 3{\cdot}2^2} = 2{\cdot}4$ m.
Resolving tension along the horizontal and vertical
$T\sin\theta =$ centripetal force $= m\,\omega^2 r = {\cdot}06 \times \omega^2 \times 2{\cdot}4$
$T\cos\theta = {\cdot}06g$ (as there is no vertical acceleration)

$\therefore \quad T \times \cos 37° = {\cdot}06g$

or $\quad T \times {\cdot}8 = {\cdot}06g$ or $T = {\cdot}075g = {\cdot}075$ kgf. *Ans.*

$\therefore \quad {\cdot}075g \times \sin 37° = {\cdot}06 \times \omega^2 \times 2{\cdot}4$

or $\quad {\cdot}075g \times {\cdot}6 = {\cdot}06 \times \omega^2 \times 2{\cdot}4$ or $\omega = 1{\cdot}75$ rad/s.

$$\tau = \frac{2\pi}{\omega} = \frac{2\pi}{1{\cdot}75} = 3{\cdot}6 \text{ s.} \quad Ans.$$

7. Let x be the distance by which it goes down. Taking the point of suspension as reference level we have from energy conservation

$0 - mgl = \frac{1}{2}kx^2 - mg(l+x)$ where $l =$ unstretched length of the spring

or
$$kx^2 = 2mgx \quad \text{or} \quad x = \frac{2mg}{k}$$

$$= \frac{2 \times {\cdot}2 \times 9{\cdot}8}{19{\cdot}6} = 0{\cdot}2 \text{ m.}$$

In the equilibrium position, that is, if vibrations are stopped let the spring elongate by x_0. Then $kx_0=mg$

or $$x_0=\frac{mg}{k}=0\cdot 1 \text{ m};$$

$\therefore$ amplitude $=x-x_0=\cdot 2-\cdot 1=\cdot 1$ m. *Ans.*

$$\text{time period}=2\pi\sqrt{\frac{m}{k}}=2\pi\sqrt{\frac{\cdot 2}{19\cdot 6}}=\cdot 63 \text{ s}$$

and $$\text{frequency } (f)=\frac{1}{T}=\frac{1}{6\cdot 3}=1\cdot 58 \text{ Hz.}$$

$$U\,(\text{energy})=\tfrac{1}{2}ma^2\omega^2$$
$$=\tfrac{1}{2}\times 2\times \cdot 1^2\times\left(\frac{2\pi}{\cdot 63}\right)^2$$
$$=\cdot 1 \text{ joule.} \quad \textit{Ans.}$$

8. (a) The radius of the uniform circular motion is obviously the length of the cord. Let it be l.

$$\text{Centripetal force}=k(l-l_0)$$
$$=m\omega^2 l$$
$$=m4\pi^2f^2l \qquad (\because \ \omega=2\pi f)$$

or $$l=\frac{kl_0}{k-4\pi^2mf^2}. \quad \textit{Ans.}$$

(b) $$T\ (\text{tension of the cord})=k(l-l_6)$$
$$=k\left[\frac{kl_0}{k-4\pi^2mf^2}-l_0\right]$$
$$=\frac{4\pi^2mf^2kl_0}{k-4\pi^2mf^2}. \quad \textit{Ans.}$$

9. Let A be the point of suspension and N be the position of the nail. Then

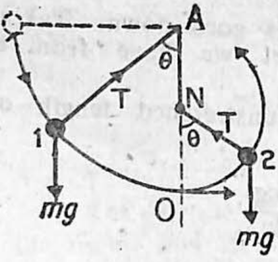


Fig. 5.64

$$AN=\frac{l}{2}.$$

From O' to O the trajectory is a circle of radius l and beyond O, it is a circle of radius $\frac{l}{2}$. Let us consider two positions of the ball. For position 1 we have

$$T-mg\cos\theta=\frac{mv^2}{l}, \text{ by dynamics of circular motion}$$

and $mgl=\frac{1}{2}mv^2+mg(l-l\cos\theta)$, from enregy consideration

or $$mgl\cos\theta=\frac{1}{2}mv^2$$

or $$v^2=2gl\cos\theta.$$

$$\therefore \quad T=mg\cos\theta+2mg\cos\theta=3mg\cos\theta.$$

For none of the values of θ from O to $\pi/2$, cosθ is zero and so in the part O' to O, tension does not disappear. In the position 2

$$T-mg\cos\theta=m\times\frac{v^2}{\frac{l}{2}}=\frac{2mv^2}{l}$$

and $$mgl=\frac{1}{2}mv^2+mg\left(\frac{l}{2}-\frac{l}{2}\cos\theta\right)$$

or $$\frac{mgl}{2}=\frac{1}{2}mv^2-mg\frac{l}{2}\cos\theta$$

or $$v^2=gl(1+\cos\theta)$$

$$\therefore \quad T-mg\cos\theta=\frac{2mgl(1+\cos\theta)}{l}$$

or $$T=2mg+3mg\cos\theta.$$

When $T=0$, $$2mg+3mg\cos\theta=0$$

or $$\cos\theta=-\tfrac{2}{3}$$

or $$\cos(180^\circ-\theta)=\tfrac{2}{3}=\cos 48^\circ$$

$\therefore$ $\theta=132^\circ$ or 42° above the horizontal line. *Ans.*

LESSON 6

Topics: Gravitation
Simple Pendulum
Elasticity

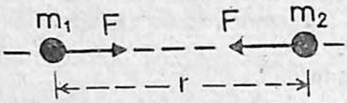
6.1 Gravitation

Newtons law of gravitation.

Two point masses m_1 and m_2, separated by a distance r attract each other with a force

$$F=G\frac{m_1m_2}{r^2}$$

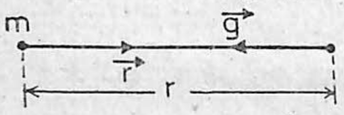
where $G=6{\cdot}67\times10^{-11}\ \frac{Nm^2}{kg^2}$ = Universal Constant of Gravitation.



This force between two masses acts equally on both masses, though in opposite directions. It does not depend on the medium present between the two masses.

● **Gravitational field** : This is a region in space where any mass will experience a force. The gravitational field strength (g) at a point is the force acting on unit mass placed at the point. It is a vector.

Any mass sets up a gravitational field around it. The gravitational field strength at a distance r from a point mass m is



$$g=-G\frac{m}{r^2}.$$

The $-$ve sign emphasizes the fact that gravitational field is always attractive.

● **Gravitational potential** : The gravitational potential (V) at a point is the work that has to be done to bring unit mass from infinity to that point. It is a scalar. The gravitional potential at a distance r from a point mass m is

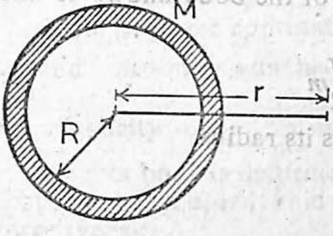
$$V=-G\frac{m}{r}.$$

The negative sign arises because in bringing the unit mass from infinity, work is done by the system, so that its potential energy decreases.

The potential at a point does not depend on the actual path followed in bringing the unit mass from infinity. Thus, gravitational force is a conservative force.

Gravitational field (g) and potential (V) due to a spherical shell and solid sphere :

Shell



(i) Outside

$$g=-G\frac{M}{r^2}$$

$$\vec{g}=-G\frac{M}{r^2}\,\hat{r}$$

$$V=-G\frac{M}{r}$$

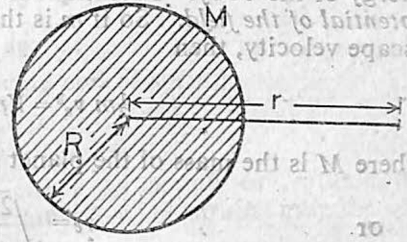
(ii) Inside

$$g=0$$

$$V=-G\frac{M}{R}$$

where R is the radius of the shell.

Sphere

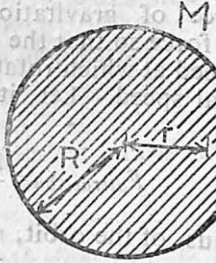


(i) Outside

$$g=-G\frac{M}{r^2}$$

$$\vec{g}=-G\frac{M}{r^2}\,\hat{r}$$

$$V=-G\frac{M}{r}$$



(ii) Inside

$$g=-G\frac{M}{R^3}\,r$$

where R is the radius of the sphere.

$$V=-G\frac{M}{2R^3}(3R^2-r^2).$$

● **Escape Velocity.** The minimum velocity to be imparted to a body on the surface of a planet, so that it is carried beyond the gravitational field of that planet is called the escape velocity of that planet. Obviously to carry the body beyond the gravitational field, that amount of energy is needed which is required to bring it from infinity up to the surface of the planet. This is exactly the potential energy of the body. *Potential energy per unit mass is equal to the potential of the field.* So if m is the mass of the body and v_e is the escape velocity, then

$$\tfrac{1}{2}m\,v_e^3 = Vm = G\frac{M}{R}m$$

where M is the mass of the planet and R is its radius.

or
$$v_e = \sqrt{\frac{2GM}{R}}.$$

If g is the gravitational field intensity, then

$$mg = \frac{GMm}{R^2}$$

or
$$GM = gR^2.$$

Note: This formula $GM = gR^2$ is of frequent use to find value of G from g. Commit it to memory.

$$\therefore \quad v_e = \sqrt{2gR}.$$

● **Satellites and Orbital Speed** (v_0). A satellite is a small body revolving around a large body under the gravitational attraction of the latter. The force of gravitational attraction provides the necessary centripetal force so that the satellite may be in rotational equilibrium. The speed at which rotational equilibrium is attained is called the orbital speed. Let it be v_0. Then for rotational equilibrium we have

$$F_{attraction} = \frac{mv_0^2}{r}$$

where r is the radius of the orbit, measured from the centre of the planet.

or
$$G\frac{Mm}{r^2} = \frac{mv_0^2}{r}$$

or
$$v_0 = \sqrt{\frac{GM}{r}}.$$

6.2 Pendulums

The period of a simple pendulum for small oscillations is

$$T = 2\pi\sqrt{\frac{l}{g}}$$

where l = effective length of the pendulum = distance between point of support and centre of gravity of bob, and g = acceleration due to gravity.

The period of an oscillating spring

$$T=2\pi\sqrt{\frac{m}{k}}$$

where k = force constant of the spring

and m = mass attached to moving end of spring.

6.3 Elasticity

When a body is deformed, i.e., either its length or volume or shape is changed, it is said to be 'strained'. The strain may be of three types :

$$\text{Longitudinal strain}=\frac{\text{change in length}}{\text{original length}}$$

$$\text{Volume strain}=\frac{\text{change in volume}}{\text{original volume}}.$$

Shear strain is the angle through which one edge of the body is deflected, when a force is applied perpendicular to that edge.

Thus strain is a dimensionless quantity. When a body is strained, its elasticity creates forces of reaction inside it which tend to restore it to its original (unstrained) state. The force of reaction per unit area is called "stress". Thus, 'strain causes stress'. Stress has the same dimension as pressure.

Hooke's law and relation of stress and strain :

$$\frac{\text{stress }(\sigma)}{\text{strain }(\epsilon)}=\text{a constant called modulus of elasticity.}$$

That is, stress $\propto$ strain.

It follows from this law that *elongation is proportional to load.*

$$\text{Young's modulus }(Y)=\frac{\text{longitudinal stress}}{\text{longitudinal strain}}.$$

$$\text{Bulk modulus }(K)=\frac{\text{volume stress}}{\text{volume strain}}.$$

$$\text{Rigidity or modulus of shear strain }(n)=\frac{\text{shear stress}}{\text{shear strain}}.$$

Note. In case of gas or liquid, pressure is the 'stress' on the system.

Strain potential energy :

Strain potential energy per unit volume = strain P.E. density (u) $= \frac{1}{2} \times$ stress $\times$ strain.

Work done by a force (F) in stretching a wire through $\triangle l = \frac{1}{2} \times F \times \triangle l$.

Spring force and potential energy of a spring :

$$F_{spring} = kx$$

where x = change in length of the spring, and k is a constant called 'force constant' of the spring. It depends on the length, material and area of the spring.

$$F_{spring} = \frac{\lambda x}{l}$$

where λ is a constant depending on the material and area of the spring, called 'modulus' of the spring.

Potential energy of a strained spring $= \frac{1}{2} kx^2$.

Note The absolute unit of stress is N/m^2. It is also called a pascal (Pa). A GPa (giga pascal) is $10^9 N/m^2$.

Thermal expansion or contraction of a wire or rod $= l\alpha \triangle \theta$ where l = original length, α = coefficient of linear expansion and $\triangle \theta$ = change in temperature.

CRITICAL QUESTIONS

1. If the force of gravity acts on all bodies in proportion to their masses, why does not a heavy body fall faster than a rigid light body ?

Ans. Though heavier bodies are acted upon by greater forces, the acceleration produced in them is the same as that produced in light bodies. Let F be the force on any body of mass m and g be the acceleration. Then

$$F = G \frac{Mm}{R^2} = mg \text{ or } g = \frac{GM}{R^2},$$ an expression independent of mass.

Hence all bodies fall with the same rapidity provided there is no air resistance.

2. Does the concentration of the earth's mass near its centre change the variation of g (acceleration due to gravity) with height from its surface ?

Ans. No. The acceleration due to gravity, that is, gravitational intensity at points outside the surface may be calculated by assuming the mass to be concentrated at the centre of the sphere. Hence whether the earth were a homogeneous sphere or a concentrated mass system, the variation of g with height is the same.

3. Why can we learn more about the shape of the earth by studying the motion of an artificial satellite than by studying the motion of the moon ?

Ans. We can learn more about the shape of the earth by studying the motion of an artificial satellite than by studying the motion of the moon because the motion of the satellite is governed at all times by the value of *g* at its position. Since the moon is very far away, the variations of *g* near the surface of the earth are averaged out at the position of the moon.

4. One clock is based on an oscillating spring, the other on the pendulum. Both are taken to Mars. Will they keep the same time there as they kept on Earth ? Will they agree with each other ? Explain.

Ans. The spring-operated clock will keep the same time but the pendulum clock will not keep the correct time as *g* on Mars is different from *g* on Earth. They will not agree.

5. The gravitational force exerted by the Sun on the Moon is almost twice as great as the gravitational force exerted by the Earth on the Moon. Why then doesn't the Moon escape from the Earth ?

Ans. Because of rotational equilibrium of the moon round the earth and also that of the earth + moon system round the sun. The attractive force on the moon due to earth provides necessary centripetal force for rotational motion of the moon round the earth and the attractive force on the moon due to the sun provides necessary centripetal force for rotational motion of the moon round the sun.

6. Would you expect the total energy of solar system to be constant ? The total angular momentum ? Explain.

Ans. Yes. The gravitational forces are conservative. When a system is under the action of conservative forces, its total energy (kinetic + potential) is a constant and hence energy of the solar system is a constant. The gravitational forces on planets of the solar system are central forces directed to the sun and hence moment of forces about the sun is zero. Thus no torque acts on the planets and so angular momentum is a constant.

7. Objects at rest on the earth's surface move in circular paths with a period of 24 hours. Are they in 'orbit' in the sense that an earth satellite is in orbit ? Explain.

Ans. No, they are not in 'orbit' in the sense that an artificial satellite is, because the whole of the gravitational pull on the artificial satellites provides the necessary centripetal force but for objects at rest on the earth whole of their gravitational pull (i e. their weights) does not provide the necessary centripetal force; only a small fraction provides the necessary centripetal force. In short, bodies in 'orbit' are weightless but bodies on the earth at rest are not weightless.

8. The sun's tide-raising power is only half as great as that of the moon's. The direct pull of the sun on the earth, however, is about

175 times that of the moon. Why is it then that the moon causes the larger tides ?

Ans. The tide-raising power of a body (sun or moon) is determined by the 'difference in gravitational attraction' between the far and near sides water, not by the total pull. The tide producing force varies inversely as the cube of the distance. Thus the relatively short distance separating the earth and the moon makes the moon's tide-raising force more than double that of the distant sun.

9. Does a rocket really need the escape velocity from the very beginning to escape from the earth ?

Ans. No. What is actually needed is the supply of total amount of energy required to take it beyond the gravitational field of the earth.

10. Can a satellite move in a stable orbit in a plane not passing through the earth's centre ? Explain.

Ans. No. For stability of rotational motion there should not be any torque of the force acting on it about the centre of the orbit. When the orbit does not lie in a plane passing through the centre, its gravitational pull will have moment (torque) about the centre of the orbit and so it cannot be in stable orbit.

11. The total energy of the earth+sun system is negative. How do you interpret the negative energy of a system ?

Ans. The meaning of the total energy being negative is that the system (sun+earth) is a closed one, the earth always being bound to the attracting solar centre and never escaping from it.

12. As measured by an observer on earth would there be any difference in the periods of two satellites, each in a circular orbit near the earth in an equatotial plane, but one moving eastward and the other westward ?

Ans. The westward satellite will appear to have shorter period than the eastward satellite, owing to the rotation of the earth itself from west to east.

13. An artificial satellite in the presence of frictional forces will move into an orbit closer to the earth and may have increased kinetic energy. Explain this.

Ans. We have E (total energy) $=\frac{1}{2}mv^2+\left(-\frac{GMm}{r}\right)$ where m is the mass of the satellite, $M=$ mass of the earth and $r=$ radius of the orbit.

By 'dynamics' of circular motion

$$\frac{GMm}{r^2}=\frac{mv^2}{r} \quad \text{or} \quad v^2=\frac{GM}{r}$$

$$\text{or} \qquad E=-\frac{1}{2}\frac{GM}{r} \quad \text{or} \quad -\frac{1}{2}mv^2 \quad \text{or} \quad -K \text{ (kinetic energy).}$$

$\therefore$ E_i (initial energy) $= -\frac{1}{2}\frac{GM}{r_0}$ and E_f (final energy) $= -\frac{1}{2}\frac{GM}{r}$.

By 'energy conservation' principle

$$E_i = E_f + W \text{ (work done by frictional force)}$$

$$\therefore \quad -\tfrac{1}{2}\frac{GM}{r_0} = -\tfrac{1}{2}\frac{GM}{r} + W$$

or $$\tfrac{1}{2}GM\left(\frac{1}{r} - \frac{1}{r_0}\right) = W \text{ (a positive quantity)}$$

$$\therefore \quad \frac{1}{r} > \frac{1}{r_0} \quad \text{or } r < r_0.$$

Also $$-K_0 = -K + W$$

or $$K - K_0 = W \text{ (a positive quantity)}$$

$$\therefore \quad K > K_0.$$

14. If a planet of given density were made larger, its force of attraction for an object on its surface would increase because of the planet's greater mass but would decrease because of the greater distance from the object to the centre of the planet. Which effect predominates ?

Ans. Mass-effect dominates over the distance-effect [Mass $\propto R^3$, force of attraction $\propto \frac{1}{R^2}$].

15. A stone is dropped along the centre of a deep vertical mine shaft. Assume no air resistance but consider earth's rotation. Will the stone continue along the centre of the shaft ? If not, describe the motion.

Ans. Due to rotation of the earth from west to east the stone is deviated slightly to the east.

16. Two air bubbles with radius r are present in water. Are these bubbles attracted or repelled ?

Ans. The two bubbles attract each other. A body immersed in an extended denser medium behaves like a negative mass so far gravitational attraction is concerned. The two bubbles, therefore, behave like two neagative masses, each of absolute value equal to the mass of equal volume of water. So by the law gravitation

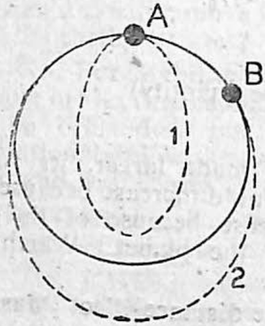
$$\vec{F} = -G\frac{m_1 m_2}{r}\hat{r} \text{ we have } \vec{F} = -G\frac{(-m_1)(-m_2)}{r^2}\hat{r}$$

$$= -G\frac{m_1 m_2}{r^2}\hat{r}.$$

The negative force of interaction means force of attraction.

17. Two satellites move along a circular orbit in the same direction at a small distance from each other. A container has to be thrown from the first satellite onto the second one. When will the container reach the second satellite earlier : if it is thrown in the direction of motion of the first satellite or in the opposite direction? The velocity of container is small in comparison to that of the satellite.

Ans. If the container is thrown in the direction opposite to the motion of the satellite, it will begin to move along an ellipse 1 lying inside the circular orbit of the satellites. The period of revolution will be slightly less than that of satellites. Therefore, they can meet at the point of contact of the orbits (circular orbit of satellite and the elliptical orbit of the container) only after a great number of revolutions. On the other hand if the container is thrown in the direction of the satellite, the container begins to move along an ellipse 2. The velocity u should be such that during one revolution of the container, satellite B also makes one revolution and in addition covers the distance AB. So the container should be thrown in the direction of the first satellite.



18. When a train moves from west to east at high speed does its weight increase or decrease ?

Ans. The weight decreases. See Example 2.

19. Why is there 'weightlessness' in a satellite ?

Ans. The feeling of weight arises from the reaction of the ground on a man. In an orbiting satellite, the astronaut and the floor of the satellite both have the same acceleration (the centripetal acceleration towards the earth). Hence the floor of the satellite offers no reaction to the astronaut and he feels weightless.

20. Is the velocity of escape of a particle at rest on the surface of the earth the same as that of a particle just orbiting round the earth ? Explain.

Ans. No, the velocity of escape arises from the energy required to take it from the present condition to infinity. This is equal to the difference between energy on the surface of the earth and infinity. For a particle at rest

$$\tfrac{1}{2} m v_e^2 = \frac{GM}{R} m \text{ (only potential energy)}$$

$$v_e = \sqrt{\frac{2GM}{R}}.$$

For a particle in circular motion

$$E \text{ (energy)} = -\frac{GM}{R} + \tfrac{1}{2} m v_c^2.$$

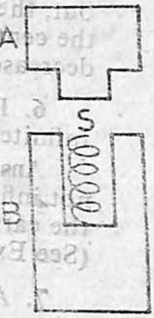
But $\frac{mv_c^2}{R} = \frac{GMm}{R^2}$, from dynamics of circular motion.

$$\therefore \quad E = -\tfrac{1}{2}\,\frac{GMm}{R},$$

$$\therefore \quad \tfrac{1}{2} m v_e^2 = \tfrac{1}{2}\,\frac{GMm}{R}$$

or $$v_e = \sqrt{\frac{GM}{R}}.$$

21. Two metal blocks are shaped such that *A* can fit into *B*. When *A* is placed on *B*, the spring *S* is compressed. The system is now allowed to fall freely under gravity. It is found that *A* and *B* separate during the fall. Explain.



Ans. Bodies falling freely under gravity are in a state of weightlessness. In this state, the compressed spring will separate *A* and *B*.

Simple Pendulum

1. Can we actually construct a simple pendulum ?

Ans. No. Because the conditions imposed on a simple pendulum such as weightless string, perfectly flexible, inextensible string and perfectly rigid support can never be realised in practice.

2. Could standards of mass, length and time be based on properties of a pendulum ? Explain.

Ans. No. Because time period of a pendulum varies from place to place. Anything which is not fixed cannot be used as a standard. Even if the period and length of a pendulum at a given place can perhaps be used as standard of time length respectively, its mass cannot be used as standard of mass because any mass suspended will have the same period.

3. How is the period of a simple pendulum affected when its point of suspension is moved horizontally with acceleration '*a*' ?

Ans. The period of pendulum is decreased. The exact expression for *T* is

$$T = 2\pi \sqrt{\frac{l}{\sqrt{a^2+g^2}}}.$$ For deduction see Introductory Physics Part I, Chapter 10.

4. A pendulum is mounted on a cart rolling down a smooth inclined plane. In what way will the pendulum rest with respect to an observer in the cart and what will be its time period ?

Ans. The pendulum will lie perpendicular to the floor of the cart relative to an observer in the cart. The period of the pendulum is given by the formula given in Ex. 2.

5. A hollow sphere, filled with water, has a small hole in it. It is hung by a long thread and, as water slowly flows out of the hole at the bottom, one finds that the period of oscillation first increases and then decreases. Explain.

Ans. The centre of gravity goes below the centre of the sphere as water flows out. This causes increase in the length of the pendulum and so its period increases. When all the water flows out, the centre of gravity of the hollow sphere again comes back to the centre of the sphere and so the effective length of the pendulum decreases and consequently its period also decreases.

6. Is the time period of a simple pendulum of infinite length infinite ? If not, what is the upper limit ?

Ans. The time period of a simple pendulum of infinite length is not infinite but has a moderate value determined by the radius of the earth and acceleration due to gravity. Its value is 84·3 min (See Example 1).

7. A big hollow sphere has a large cavity off the centre of the sphere, almost touching its surface internally. It is sealed from all sides and looks identical in all positions. Can you find the point where the cavity touches the sphere by using it as the bob of a pendulum ?

Ans. Yes. The point about which the simple pendulum set up with this hollowed sphere as bob has minimum time period is the required point.

8. A pendulum is constructed by suspending a thin-walled sphere filled with water by a rigid light rod. Will the time period change if the water in the sphere freezes ? The viscosity of water and the change in volume on freezing are negligible.

Ans. Yes. The time period will increase. Water in the liquid state moves translationwise and so the system forms an ideal or a mathematical pendulum. When water freezes into ice, it behaves like a solid mass having rotational motion as well. Hence the whole system (rigid rod+rigid sphere) forms a compound pendulum. A compound pendulum has greater time period. So the period will increase.

Elasticity

1. Stress and pressure are both forces per unit area. Then in what respect stress differs from pressure ?

Ans. Pressure is the external force per unit area but stress is the internal force that is called into play from within a strained body per unit transverse area of it.

2. Is elastic limit a property of the material of the wire ?

Ans. No. It depends on the radius of the wire as well.

3. Which is more elastic, steel or rubber ?

Ans. Steel. Elasticity is measured by the stress per unit strain. For the same amount of strain, much greater stress is produced in steel than in rubber.

4. Which one has got greater force constant—steel wire or a rubber wire of the same length and radius ?

Ans. The steel wire has got greater force constant.

5. There are two springs of the same material and length but area of cross-section of one is double that of the other. Which one will have greater force constant ?

Ans. The one having greater area will have less force constant.

6. Among solids, liquids and gases which one can have all the modulii of elasticity ?

Ans. Only solids. Liquids and gases have only bulk modulus.

7. Among solids, liquids and gases which one possesses greatest bulk modulus ?

Ans. Solids.

8. In the dermination of Young's modulus of the material of wire by Vernier method two long wires of the same material are suspended from a rigid support. Explain, why ?

Ans. The lengths are kept large so that elongation may be sufficiently large to be accurately measured. Because of large lengths the change in length due to temperature fluctuations will not be negligible. Hence two wires are used so that thermal expansion for both may be the same and this may not be mixed up with the elastic expansion.

9. A copper plate is soldered to two steel plates. All the plates have the same cross-section. Will tensions be same in all the plates if the system is uniformly heated ?

Ans No. Since copper expands more than steel, increase of temperature will lead to compression of the copper plate and tension in the steel ones. If T is the mutual force exerted by copper plate on the adjoining steel plate, then tension of each plate will be T but tension in copper plate will be $2T$ because it is acted upon by both steel plates from the two sides. Thus tension in copper plate is double the tension in the either steel plate.

10. Why are boilers shaped spherical or cylindrical ?

Ans. A spherical boiler can withstand greater pressure than boiler of any other shape. See mathematical proof in Ex. 8.

11. Why is the surface of a liquid in earth's gravitational field always horizontal ?

Ans. A liquid cannot sustain shearing stress and so its surface always lies perpendicular to the force to which it is subjected. *Remember shearing force parallel to a liquid surface is always zero.* A liquid is subjected to a vertical force in the earth's gravitational field and so its surface is always horizontal.

12. When a steam boiler explodes there is excessive damage but when a cylinder filled with water and pressed to the same pressure as that of steam explodes, no serious damage is caused. Why ?

Ans. This is due to very high compressibility of steam. When a boiler explodes steam tends to gain its original volume violently because of its high compressibility and so heavy damage is caused by it. But water is almost incompressible and so when its container explodes, nothing serious happens.

OBJECTIVE QUESTIONS

Gravitation

1. The height at which the acceleration due to gravity would be reduced to a value one-fourth of that at the surface of the earth is
(a) $\frac{1}{2}R$, (b) $\frac{3}{4}R$, (c) $\frac{1}{4}R$, (d) R.

Ans. (d). We have $GM=gR^2=g'(R+h)^2$.
Here $g'=g/4$ $\therefore$ $gR^2=g/4(R+h)^2$ or $R=\frac{1}{2}(R+h)$ or $h=R$.

2. The horizontal speed with which a satellite must be projected at 316 km above the surface of the earth so that it will have a circular orbit about the earth ($R=6400$ km, $g=10$ ms^{-2}) is
(a) 7500 m/s, (b) 7800 m/s, (c) 8000 m/s, (d) 3200 m/s.

Ans. (b). We have $\frac{mv_0^2}{(R+h)}=\frac{GMm}{(R+h)^2}$

or $$v_0^2=\frac{gR^2}{R+h} \quad (\because \quad GM=gR^2)$$

$$=gR\left(1+\frac{h}{R}\right)^{-1}=g\,(R-h) \quad (\because \quad h<<R)$$

$$\therefore \quad v_0=\sqrt{g\,(R-h)}=\sqrt{10\times(6400-316)\times10^3}=10^2\sqrt{6084}$$
$$=10^2\times78=7800 \text{ m/s}.$$

3. The velocity of escape from the surface of the earth ($R=6400$ km and $g=10$ ms^{-2}) is (a) 5 km/s, (b) $8\sqrt{3}$ km/s, (c) $8\sqrt{2}$ km/s, (d) $4\sqrt{2}$ km/s.

Ans. (c). $\frac{1}{2}mv_e^2 = \frac{GMm}{R} = \frac{mgR^2}{R} = mgR$

$$\therefore \quad V_e = \sqrt{2gR} = \sqrt{2\times 10\times 6400\times 10^3} = 8\sqrt{2}\times 10^3 \text{ m/s}$$
$$= 8\sqrt{2} \text{ km/s.}$$

4. The ratio of the energy required to raise a satellite up to a height h above the earth to the one required to put it in orbit there is (a) $h : R$, (b) $R : h$, (c) $2h : R$, (d) $R : 2h$.

Ans. (c). Work done (W) to take it to height h = energy (E_1) required = change in potential energy

$$= \left(-\frac{GMm}{R+h}\right) - \left(-\frac{GMm}{R}\right) = GMm\frac{h}{R(R+h)}$$

or $$E_1 = gR^2m\frac{h}{R^2\left(1+\frac{h}{R}\right)} = \frac{mgh}{\left(1+\frac{h}{R}\right)}.$$

E_2 = energy required to put in orbit $= \frac{1}{2}mv_0^2$

(because energy of a body in orbit is $\frac{1}{2}mv_0^2$)

$$\frac{mv_0^2}{R+h} = \frac{GMm}{(R+h)^2} \text{ or } v_0^2 = \frac{gR^2}{R+h} = \frac{gR}{1+\frac{h}{R}}$$

$$\therefore \quad E_2 = \frac{1}{2}m.\frac{gR}{1+\frac{h}{R}} = \frac{mgR}{2\left(1+\frac{h}{R}\right)}.$$

$$\therefore \quad \frac{E_1}{E_2} = \frac{2h}{R}. \quad \therefore \; E_1 : E_2 = 2h : R.$$

5. The angular velocity of the earth about its axis (take a day to be of 24 hours duration on average) is

(a) $7{\cdot}27\times 10^{-5}$ rad/s, (b) $6{\cdot}0\times 10^{-9}$ rad/s, (c) $4{\cdot}2\times 10^{-6}$ rad/s, (d) $7{\cdot}17\times 10^{-7}$ rad/s.

Ans. (a).

Note. The angular velocity of the earth about its axis is a constant and its value is required in many problems. Commit it to memory.

6. The potential energy of the gravitational interaction between two point masses m_1 and m_2 located r distance from each other is

(a) $G\frac{m_1m_2}{r^2}$, (b) $-G\frac{m_1m_2}{r}$, (c) $G\frac{m_1m_2}{r}$, (d) $G\frac{m_1+m_2}{r}$.

Ans. (b).

7. A spaceship is launched into a circular orbit close to the earth's surface. The additional velocity to be imparted to the spaceship to drive it out of gravitational field is

(a) $gR(\sqrt{2}-1)$, (b) $\sqrt{gR}\,(\sqrt{2}-1)$, (c) $\sqrt{2gR}$, (d) $\sqrt{gR}$ where R is the radius of the earth and g = acceleration due to gravity.

Ans. (b). We have v_e (escape velocity) $=\sqrt{2gR}$ and v_o (orbital velocity) $=\sqrt{gR}$. Therefore, the additional velocity required is the difference of the two, that is, $\sqrt{2gR}-\sqrt{gR}=\sqrt{gR}\,(\sqrt{2}-1)$.

8. A satellite of mass 200 kg revolves around a planet of mass 5×10^{30} kg in a circular orbit $6{\cdot}6\times10^{6}$m. The binding energy of the satellite ($G=6{\cdot}6\times10^{-11}$) is

(a) $+5\times10^{3}$ J, (b) -10^{5} J, (c) $+5\times10^{15}$ J, (d) -5×10^{15} J.

Ans. (d). U_b (binding energy)

$$=-\tfrac{1}{2}\,mv^2 \text{ or } -\tfrac{1}{2}\frac{GMm}{r}.$$

$$\therefore \quad U_b=-\tfrac{1}{2}\frac{6{\cdot}6\times10^{-11}\times5\times10^{30}\times200}{6{\cdot}6\times10^{-6}}=-5\times10^{15}\text{ J}.$$

9. If the radius of the earth were to shrink by one per cent its mass remaining the same, the acceleration due to gravity on the earth's surface would (a) decrease, (b) increase, (c) remain unchanged. **(I. I. T. 1981)**

Ans. (b). We have $GM=gR^2$. $\therefore$ gR^2 = a constant.

$$\therefore \quad g\propto\frac{1}{R^2}.$$

10. If v_e is velocity of escape of a particle at rest on the surface of the earth and V_e that of a particle in circular motion round the earth, then

(a) $v_e=V_e$, (b) $v_e=2V_e$, (c) $v_e=\sqrt{2}V_e$, (d) $V_e=\sqrt{2}V_e$.

Ans. (c).

Pendulum

1. The time period of a pendulum of length 1 m at a place where $g=10$ m/s^2 is (take $\pi^2=10$) (a) 1·5 s, (b) 3 s, (c) 2 s, (d) 1 s.

Ans. (c).

2. The length of a seconds pendulum at a place where $g=10$ m/s^2 is (take $\pi^2=10$) (a) 1·5 m, (b) 1 m, (c) 0·9 m, (d) 0·5 m.

Ans. (b).

3. The period of a simple pendulum of infinite length is

(a) infinite, (b) zero, (c) same as the time of revolution of a satellite just above the earth's surface, (d) exactly 2 s.

Ans. (c). See Example 1.

4. A simple pendulum hangs from the an elevator cabin. If the cabin moves up with uniform acceleration 'a', the period of oscillation of the pendulum compares to the period when the cabin is at rest as (a) $\sqrt{g} : \sqrt{a}$, (b) $\sqrt{g+a} : \sqrt{g}$, (c) $g : a$, (d) $\sqrt{g} : \sqrt{g+a}$.

Ans. (d). When at rest, $T = 2\pi\sqrt{\frac{l}{g}}$. When there is uniform upward acceleration, the effective acceleration is $g+a$.

$$\therefore \quad T' = 2\pi\sqrt{\frac{l}{g+a}}.$$

$$\frac{T'}{T} = \sqrt{\frac{g}{g+a}}.$$

5. A simple pendulum gains 4 seconds per minute at a place but loses 12 seconds per minute at another place. The acceleration due to gravity at the first place compares to that of the second place as (a) 3 : 4, (b) 4 : 3, (c) 16 : 9, (d) 9 : 16.

Ans. (c). $2\times\frac{60}{60+4} = 2\pi\sqrt{\frac{l}{g_1}}$ and $2\times\frac{60}{60-12} = 2\pi\sqrt{\frac{l}{g_2}}$

Dividing, $\frac{48}{64} = \sqrt{\frac{g_2}{g_1}}$ or $\frac{g_1}{g_2} = \frac{16}{9}$.

6. A simple pendulum is suspended from the ceiling of the cabin of a sky laboratory. The man in the laboratory pushes the bob to one side. (a) The bob remains stationary, (b) the bob rotates continuously about the point of suspension with the velocity imparted to it by the man, (c) it oscillates with a very large period, (d) it vibrates very rapidly.

Ans. (b). Since there is 'weightlessness' in the sky laboratory, there will be no restoring force on the bob. In the absence of any restoring force, the bob will rotate with the velocity imparted to it due to inertia.

Elasticity

1. A wire 2 m long, ·05 cm² in cross-section, when stretched by a weight of 2 kg increases in length by ·04 mm. The Young's modulus of the material of the wire is ($g = 10$ ms^{-2})
(a) 20×10^9 N/m², (b) 20×10^{10} N/m², (c) 20×10^{11} N/m², (d) 2×10^8 N/m².

Ans. (b).

2. In the above example the work done in stretching the wire is (a) 2×10^{-3}J, (b) 3×10^{-4} J, (c) 4×10^{-4} J, (d) 5×10^{-4} J.

Ans. (c). $W=\frac{1}{2}\times F\times \triangle l=\frac{1}{2}\times 2\times 10\times(\cdot 04\times 10^{-3})=4\times 10^{-4}$ J.

3. If ρ is the density of the material of a wire and σ is the breaking stress, the greatest length of the wire that can hang freely without breaking is

(a) $\frac{\sigma}{\rho g}$, (b) $\frac{\sigma}{2g\rho}$, (c) $\frac{2\sigma}{\rho g}$, (d) $\frac{\rho}{2\sigma g}$.

Ans. (a). $\text{stress}=\frac{\text{weight}}{\text{area}}=\frac{Al\rho g}{A}=l\rho g$

$$\therefore \qquad \sigma=l\rho g \quad \text{or} \quad l=\frac{\sigma}{\rho g}.$$

4. When the load on a wire is increased slowly from 2 kg to 4 kg the elongation increases from 0·6 mm to 1·0 mm. The work done during this extension of the wire is ($g=10$ m/s^2) is

(a) 14×10^{-3} J, (b) $\cdot 4\times 10^{-3}$ J, (c) 8×80^{-2} J, (d) 10^{-3} J.

Ans. (a). $W=(\frac{1}{2}F_2\times \triangle l_2)-(\frac{1}{2}F_1\times \triangle l_1)$

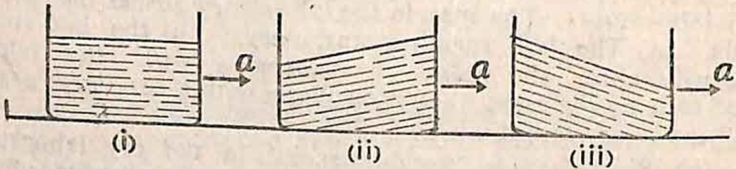
$$=(\tfrac{1}{2}\times 4\times 10\times 1{\cdot}0\times 10^{-3})-(\tfrac{1}{2}\times 2\times 10\times \cdot 6\times 10^{-3})$$
$$=20\times 10^{-3}-6\times 10^{-3}=14\times 10^{-3} \text{ J}.$$

5. A steel wire of length 1 m and density 8000 kg/m^3 is suspended from the ceiling by one of its ends. If $Y=2\times 10^{10}$ N/m^2, the elongation of the wire under its own weight ($g=10$ m/s^2) is

(a) 10^{-6} m, (b) $\frac{1}{2}\times 10^{-6}$ m, (c) 2×10^{-6} m, (d) $\frac{1}{4}\times 10^{-6}$ m.

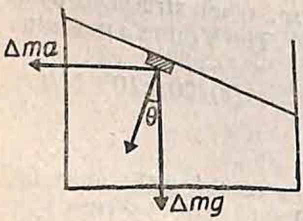
Ans. (c). See Ex. 3.

6. A vessel containing water is given a constant acceleration 'a' to the right, along a horizontal path. Which one of the diagrams represents the surface of the liquid ? **(IIT 1981)**



(i) (ii) (iii)

Ans. Let us consider an element $\triangle m$ of the liquid at the surface.



The forces to which $\triangle m$ is subjected are : (i) $\triangle mg$ downwards and (ii) $\triangle ma$ the 'inertia force' to the left. The resultant of these two forces is $\triangle m\sqrt{a^2+g^2}$, at inclination $\tan^{-1} a/g$ to the vertical to the left of it. Since shearing stress along the surface has to be zero, the surface lies as shown in the figure.

EXAMPLES

Gravitation

1. Two masses M_1 and M_2 at infinite distance between them and initially at rest, start interacting gravitationally. Find their velocity of approach when they are at a distance s apart.

Sol. Since they move under their mutual attraction and no external force acts on them, their momentum and energy are conserved.

$\therefore \quad 0=\frac{1}{2}M_1v_1^2+\frac{1}{2}M_2v_2^2-\dfrac{GM_1M_2}{s}$ (it is zero because in the beginning both kinetic energy and potential energy are zero)

$$0=M_1v_1+M_2v_2.$$

Solving the equations we have $v_1^2=\dfrac{2GM_2^2}{s(M_1+M_2)}$

and $\qquad v_2=\dfrac{2GM_1^2}{s(M_1+M_2)}$

V (velocity of approach) $=v_1-(-v_2)=v_1+v_2$

$$=\sqrt{\frac{2G(M_1+M_2)}{s}}.\ \textbf{Ans.}$$

*2. Calculate the apparent weight of a body of mass m at a latitude λ when it is moving with speed v on the surface of the earth from west to east at the same latitude.

Sol. The apparent weight (W) of a body on the surface is the 'vector difference' between the true weight mg and the centripetal force. At a latitude λ the body moves in a circle of radius $R\cos\lambda$ with angular speed ω equal to that of the earth about its axis. The modulus of the 'vector difference' of this force and mg (towards the centre) is the apparent weight. Remembering that the angle between these two forces is λ, we have

$$W^2=m^2g^2+(m\omega^2R\cos\lambda)^2-2mg\,(m\omega^2R\cos\lambda)\cos\lambda$$

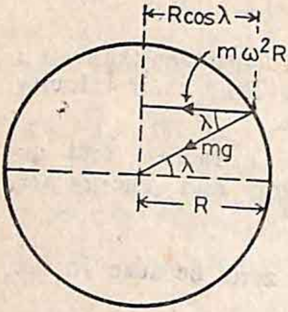
(Formula : $R^2=P^2+Q^2+2PQ\cos\theta$)

$$=m^2g^2-2m^2g\omega^2R\cos^2\lambda$$

($\because$ $m^2\omega^4R^2\cos^2\lambda$ is very small on account of fourth power of ω)

$$W=mg\left[1-\frac{2\omega^2R\cos^2\lambda}{g}\right]^{\frac{1}{2}}$$

$$=mg\left[1-\frac{\omega^2R\cos^2\lambda}{g}\right]=mg-m\omega^2R\cos^2\lambda.$$



This is the apparent weight if the body were at rest relative to the earth. Let us denote it by W_0.

$$\therefore \quad W_0 = mg - m\omega^2 R\cos^2\lambda.$$

When the body moves with speed v from west to east the angular speed becomes $\left(\omega + \dfrac{v}{R\cos\lambda}\right)$.

$$\therefore \quad W = mg - m\left(\omega + \frac{v}{R\cos\lambda}\right)^2 R\cos^2\lambda$$

$$= mg - m\left(\omega^2 + \frac{2v\omega}{R\cos\lambda}\right) R\cos^2\lambda \text{ (neglecting smaller terms)}$$

or $$W = mg\left[1 - \frac{\omega^2 R\cos^2\lambda}{g} - \frac{2v\omega}{g}\cos\lambda\right].$$

Note. This is the most general expression for the apparent weight of any body placed anywhere on the surface of the earth.

*3. A satellite moving in circular equatorial orbit of radius $R = 2\times 10^4$ km from west to east appears over a certain point at the equator every $\tau = 11{\cdot}6$ hours. Using these data, calculate the mass of the earth. The value of $G = 6{\cdot}67\times 10^{-11}$ SI units and $R = 6400$ km.

Sol. Let ω_{earth} be the angular velocity of the earth and $\omega_{apparent}$ be the apparent angular velocity of the satellite relative to observers on the earth at the equator. If ω be its absolute angular velocity, then we have from the fact the relative velocity is the vector difference,

$$\omega_{apparent} = \omega - \omega_{earth}$$

or $$\omega = \omega_{apparent} + \omega_{earth}$$

or $$\omega = \frac{2\pi}{\tau} + \frac{2\pi}{T}$$ where τ = apparent period of the satellite

T = real period of the earth.

From dynamics of circular motion

$$\frac{GMm}{R^2} = m\omega^2 R \quad \text{or} \quad GM = \omega^2 R^3$$

or $$M = \frac{\omega^2 R^3}{G} = \frac{4\pi^2 R^3}{G}\left(\frac{1}{\tau} + \frac{1}{T}\right)^2$$

$$\therefore \quad M = \frac{4\times\pi^2\times 6{\cdot}4^3\times 10^{18}}{6{\cdot}67\times 10^{-11}}\left(\frac{1}{11{\cdot}6\times 3600} + \frac{1}{24\times 3600}\right)^2$$

$$=\frac{4\pi^2\times 6{\cdot}4^3\times 10^{25}}{6{\cdot}67\times 36^2}\left(\frac{1}{11{\cdot}6}+\frac{1}{24}\right)^2=\frac{4\pi^2\times 6{\cdot}4^3\times{\cdot}1323}{6{\cdot}67\times 36^2}$$

$=6{\cdot}0\times 10^{24}$ kg. Ans.

4. Calculate the self gravitational potential energy of matter forming (a) a thin uniform shell of mass M and radius R, (b) a uniform sphere of mass m and radius R.

Sol. In the process of formation of a body work has to be done by an external agent bit by bit to build up the body. This energy of external agent is stored as gravitational energy. This is called the self gravitational potential energy or gravitational energy of mutual gravitational interaction.

(b) Consider a sphere of any radius x.

Mass of the sphere $=\frac{4\pi}{3}x^3\rho$ where $\rho=$ density of mass.

Gravitation potential of the surface $=-G.\dfrac{\frac{4\pi}{3}x^3\rho}{x}=-\frac{4\pi}{3}G\rho\,x^2.$

This is also the work done in adding unit mass to the sphere by the external agent. When the thickness is increased by dx, mass added by the agent $(4\pi x^2 dx\rho)$.

$\therefore$ Work done by the agent in increasing the surface from x to $x+dx$

$$=\left(-\frac{4\pi}{3}G\rho x^2\right)(4\pi x^2 dx\rho)$$

$$=-\frac{16\pi^2}{3}G\rho^2x^4dx.$$

$\therefore$ Total work done by the agent $=-\dfrac{16\pi^2G\rho^2}{3}\displaystyle\int_0^R x^4dx$

$$=-\frac{16\pi^2G\rho^2R^5}{15}.$$

Now $$\rho=\frac{m}{\frac{4\pi}{3}R^3}=\frac{3m}{4\pi R^3}$$

$\therefore$ $$U\text{ (self energy)}=-\frac{16\pi^2R^5G}{15}\times\frac{9m^2}{16\pi^2R^6}=-\frac{3}{5}\frac{Gm^2}{R}.$$

(b) Consider the shell when mass m has already been piled up by the agent.

Then potential of the shell $= -\dfrac{Gm}{R}$.

This is also the work done by the agent in adding further unit mass to the shell. Therefore, the elementary work done in adding further an elementary mass dm is $-\dfrac{Gm}{R}dm$.

$\therefore$ Net work done by the agent in piling the entire mass

$$= -\int_0^m \frac{Gmdm}{R} = -\frac{Gm^2}{2R}$$

$\therefore$ U (self potential energy) $= -\frac{1}{2}\dfrac{Gm^2}{R}$.

5. Calculate the radius of the circular orbit of a stationary satellite which remains motionless with respect to the earth's surface. Also calculate its height above the surface of the earth. Radius of the earth $(R) = 6400$ km and g (acceleration due to gravity) $= 9{\cdot}8$ m/s^2.

Sol. A satellite will appear motionless when its period of revolution is the same as that of the earth, that is, $T = 24$ hours. Let r be the radius of the orbit from the centre of the earth. Then from 'dynamics of circular motion'

$$m\omega^2 r = \frac{GMm}{r^2} \text{ or, } \left(\frac{2\pi}{T}\right)^2 . r^3 = GM$$

or

$$r = \left(\frac{GMT^2}{4\pi^2}\right)^{\frac{1}{3}} = \left(\frac{gR^2T^2}{4\pi^2}\right)^{\frac{1}{3}} \qquad (\because \; GM = gR^2)$$

$$= \left(\frac{9{\cdot}8 \times 6{\cdot}4^2 \times 10^{12} \times 86400^2}{4\pi^2}\right)^{\frac{1}{3}}$$

$$= \left(\frac{9{\cdot}8 \times 6{\cdot}4^2 \times 8{\cdot}64^2}{4\pi^2 \times 10}\right)^{\frac{1}{3}} \times 10^7$$

$$= (75)^{\frac{1}{3}} \times 10^7 = 4{\cdot}2 \times 10^7 = 42 \times 10^6 \text{ m. Ans.}$$

h (height above the earth) $= r - R = 42 \times 10^6 - 6{\cdot}4 \times 10^6$
$= 35{\cdot}6 \times 10^6$m. **Ans.**

*6. Two satellites A and B of equal mass move in the equatorial plane of earth, close to the earth's surface. Satellite A moves in the same direction as that of the rotation of the earth while satellite B moves in the opposite direction. Calculate the ratio of the kinetic energy of B to that of A in the reference frame fixed to the earth.

($g = 9{\cdot}8$ m/s^2 and radius of the earth $= 6{\cdot}37 \times 10^6$ m)

Sol. Let ω_A and ω_B be the absolute angular speeds of A and B. Since they are in the same orbits their time periods must be the same, that is, $\omega_A=\omega_B$. Considering 'dynamics of circular motion' of either we have

$$m\omega_A{}^2R=\frac{GMm}{R^2}$$

or $$\omega_A{}^2R=\frac{gR^2}{R^2} \qquad (\because\ 6M=gR^2)$$

or $$\omega_A=\sqrt{\frac{g}{R}}=\omega_B$$

$$\therefore \qquad \omega_A=\omega_B=\sqrt{\frac{9\cdot 8}{6\cdot 37\times 10^6}}=1\cdot 24\times 10^{-3}=124\times 10^{-5}\ \text{rad/s.}$$

$$\omega_E=7\times 10^{-5}\ \text{rad/s.}$$

Now $\omega_{AE}=\omega_A-\omega_E=124\times 10^{-5}-7\cdot 3\times 10^{-5}=116\cdot 7\times 10^{-5}$ rad/s

ω_{BE} (relative velocity of B relative to E) $=\omega_B-(-\omega_E)$

$$=\omega_B+\omega_E$$

$$=131\cdot 3\times 10^{-5}\ \text{rad/s.}$$

$$\therefore \qquad \frac{K_B}{K_A}=\frac{\frac{1}{2}m\omega_{BE}{}^2r^2}{\frac{1}{2}m\omega_{AE}{}^2r^2}=\frac{131\cdot 3^2}{116\cdot 7^2}=1\cdot 27. \quad \textbf{Ans.}$$

*7. A pair of stars rotates about a common centre of mass. One of the stars has a mass M and that of the other is m. Their centres are a distance d apart, d being large compared to the size of either star. Derive an expression for the period of revolution of the stars about their common centre of mass. Compare their angular momenta and kinetic energies.

Sol. The centre of mass of the system divides the distance between the stars in the inverse ratio of their masses. If d_1 and d_2 are distances of M and m from centre of mass, then

$$d_1=\frac{d}{M+m}\times m \quad \text{and} \quad d_2=\frac{d}{M+m}\times M.$$

The stars will rotate in circles of radii d_1 and d_2 about their centre of mass. The same force of attraction provides the necessary centripetal force for their circular motion.

$$\therefore \qquad \frac{GmM}{d^2}=M\omega_1{}^2d_1=m\omega_2{}^2d_2$$

or $$\omega_1{}^2=\frac{Gm}{d^2d_1}=\frac{Gm}{d^2}\times\frac{M+m}{d\times m}=\frac{G(M+m)}{d^3}$$

and $$\omega_2{}^2=\frac{GM}{d^2d_2}=\frac{GM}{d^2}\times\frac{M+m}{d\times M}=\frac{G(M+m)}{d^3}$$

$$\therefore \qquad \omega_1=\omega_2=\sqrt{\frac{G(M+m)}{d^3}}. \quad \textbf{Ans.}$$

$$\frac{L_M}{L_m}=\frac{(Mv_1)d_1}{(mv_2)d_2}$$ (From the fact that moment of momentum is also angular momentum)

$$=\frac{M}{m}\times\frac{d_1^2}{d_2^2}=\frac{M}{m}\cdot\frac{m^2}{M^2}$$

or $$\frac{L_M}{L_m}=\frac{m}{M}. \quad \textbf{Ans.}$$

$$\frac{K_M}{K_m}=\frac{\frac{1}{2}Mv_1^2}{\frac{1}{2}mv_2^2}=\frac{M}{m}\cdot\frac{\omega_1^2d_1^2}{\omega_2^2d_2^2}=\frac{M}{m}\cdot\frac{d_1^2}{d_2^2} \qquad (\because\ \omega_1=\omega_2)$$

$$=\frac{M}{m}\left(\frac{m}{M}\right)^2 \quad \left(\because\ \frac{d_1}{d_2}=\frac{m}{M}\right)$$

or $$\frac{K_M}{K_m}=\frac{m}{M}. \quad \textbf{Ans.}$$

8. *A* planet of mass m moves along a circle around the sun of mass m_s with velocity $v=34{\cdot}9$ km/s with respect to the heliocentre frame of reference, that is. the sun is at the centre of the frame. Find the period of revolution of this planet around the sun and show that Kepler's third law, that is, cube of orbital radius is proportional to the square of time pevolution of planets holds. Given that $m_s=1{\cdot}97\times10^{30}$ kg and $G=6{\cdot}67\times10^{-11}$ units.

Sol. From the 'dynamics of circular motion' of the planet (assuming circular orbit) we have, $\frac{mv^2}{d}=\frac{Gmm_s}{d^2}$

or $$\frac{v^2}{d}=\frac{Gm_s}{d^2} \quad \text{or} \quad v^2=Gm_s/d.$$

$\because \quad v=\omega.d,$

$$\therefore \quad v^2=\frac{Gm_s\omega}{v} \quad \text{or} \quad \omega=\frac{v^3}{Gm_s} \quad \text{or} \quad T=\frac{2\pi\,Gm_s}{v^3}$$

Here, $$T=\frac{2\pi\times6{\cdot}67\times10^{-11}\times1{\cdot}97\times10^{30}}{(34{\cdot}9\times10^3)^3}\text{s}$$

$$=1{\cdot}19\times10^7\text{s}=\frac{1{\cdot}19\times10^7}{86400}=225 \text{ days. } \textbf{Ans.}$$

Taking the equation in terms of d and ω

$$\frac{Gmm_s}{d^2}=m\omega^2 d$$

or $$\omega^2=\frac{Gm_s}{d^3}$$

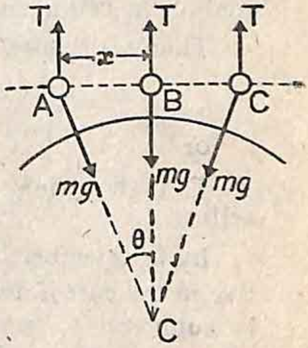
or $$\left(\frac{2\pi}{T}\right)^2=\frac{Gm_s}{d^3} \quad \text{or} \quad T^2=\frac{4\pi^2}{Gm_s}d^3$$

$\therefore$ $T^2 \propto d^3$. This is Kepler's third law.

Simple Pendulum

1. Calculate the period of a simple pendulum of infinite length and show that it is equal to the time of revolution of a satellite just above the surface of the earth.

Radius of the earth $=6400$ km and $g=10$ ms^{-2}.



Sol. Since the pendulum is of infinite length, its remains parallel to itself near the surface of the earth. Let θ be the angle made by the radius vector of the bob in one extreme position. The forces acting on the bob are (i) *mg*, towards the centre, (ii) tension *T*.

Resolving *mg* along and perpendicular to the string we have *mg* sinθ as the restoring force.

$$mg\sin\theta=-mf$$

or $$f=-g\sin\theta=-g\theta$$

$$=-g.\frac{x}{R}.$$

or $$f\propto -x,$$

$$\therefore \quad \omega^2=\frac{g}{R} \quad \text{or} \quad T=2\pi\sqrt{\frac{R}{g}}=2\pi\sqrt{\frac{6400\times 10^3}{10}}$$

$$=2\times 3{\cdot}142\times 8\times 10^2 \text{ s}$$

$$=83{\cdot}8 \text{ min}=84 \text{ min. } \textbf{Ans.}$$

Let τ be the time of revolution of a satellite near the surface of the earth.

Then $$m\omega^2 R=\frac{GMm}{R^2}$$

or $$\omega=\sqrt{\frac{GM}{R^3}}=\sqrt{\frac{gR^2}{R^3}} \quad (\because \quad GM=gR^2)$$

or $$\tau=2\pi\sqrt{\frac{R}{g}}=T. \textbf{ Proved.}$$

2. A simple pendulum of length l is suspended from the ceiling of a cart moving slowly on a horizontal level road. There is downward incline in the road forming an angle α to the horizontal. Calculate the period of oscillation of the pendulum during the motion of the cart down the incline. Assume that there is no friction.

Sol. During the motion down the plane, the cart has acceleration $g\cos\alpha$ down the plane. Let the string make angle ψ with the normal to the ceiling in the equilibrium position.

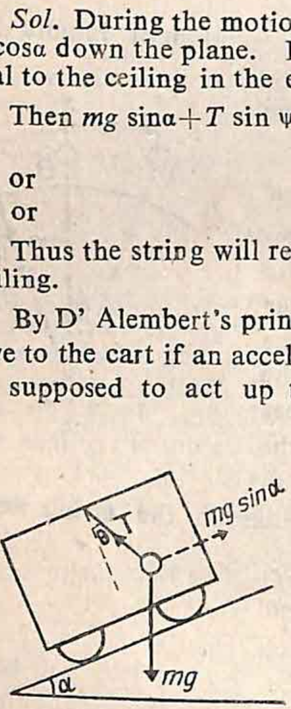
Then $mg\sin\alpha+T\sin\psi=m\times g\sin\alpha$ (considering motion of the bob down the plane)

or $$T\sin\psi=0$$

or $$\psi=0.$$

Thus the string will rest in equilibrium along the normal to the ceiling.

By D' Alembert's principle we can consider bob stationary relative to the cart if an acceleration $g\sin\alpha$ or an inertial force $mg\sin\alpha$ is supposed to act up the plane. Consider now a small displacement θ in the clockwise direction from the equilibrium position. Taking moment of all the forces acting on the bob including the 'inertia force' about point of suspension we have



$$\tau \text{ (unbalanced torque)}$$
$$=mg\ \sin\alpha l\cos\theta-mgl\sin(\alpha+\theta)$$
$$=ml^2\frac{d^2\theta}{dt^2}$$

(formula applied, torque = moment of inertia × angular acceleration)

or $$l\frac{d^2\theta}{dt}=g\sin\alpha-g(\sin\alpha+\cos\alpha.\theta)$$

$$(\because \quad \cos\theta=1 \text{ and } \sin\theta=\theta \text{ when } \theta \text{ small})$$

or $$\frac{d^2\theta}{dt^2}=-\frac{g\cos\alpha}{l}\times\theta$$

Thus the motion is simple harmonic.

$$\omega^2=\frac{g\cos\alpha}{l}$$

or
$$T=2\pi\sqrt{\frac{l}{g\cos\alpha}}.$$

3. A seconds pendulum loses 6 seconds per day on the top of a hill. Calculate the height of the hill. Radius of the earth = 6400 km and $g=10$ m/s^2 on the surface of the earth.

Sol. The pendulum makes (86400 − 6) beats in 86400 seconds and so its time period is $2\times\frac{86400}{86400-6}$

$$\therefore\quad 2\times\frac{86400}{86400-6}=2\pi\sqrt{\frac{l}{g}}.$$

and
$$2=2\pi\sqrt{\frac{l}{10}}\text{ (since the pendulum keeps correct time on the surface)}$$

$$\therefore\quad \frac{86400}{86400-6}=\sqrt{\frac{10}{g}}$$

or
$$g=10\left(1-\frac{6}{86400}\right)^2=10\left(1-\frac{1}{7200}\right).$$

But
$$g=\frac{GM}{(R+h)^2}\text{ and }10=\frac{GM}{R^2}$$

$$\therefore\quad \frac{g}{10}=\frac{R^2}{(R+h)^2}\quad\text{or } g=10\times\frac{R^2}{(R+h)^2}$$

$$\therefore\quad\left(\frac{R}{R+h}\right)^2=\left(1-\frac{1}{7200}\right)\text{ or }\frac{R+h}{R}=\left(1-\frac{1}{7200}\right)^{-2}=1+\frac{1}{3600}$$

$$\therefore\quad h=\frac{6400}{3600}\text{ km}=1{\cdot}8\text{ km. }\textbf{Ans.}$$

4. A pendulum of period 2*s* oscillates with amplitude 2 cm. The mass of the bob is 50 gm. Calculate the energy of the bob and its maximum velocity.

Sol. E energy of simple harmonic motion) $=\frac{1}{2}\,m\,a^2\omega^2$.

The bob of the pendulum is in simple harmonic motion.

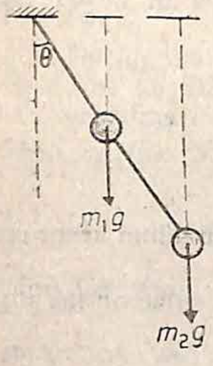
$$\therefore \quad E=\tfrac{1}{2}\times(50\times10^{-3})\times{\cdot}02^2\times\left(\frac{2\pi}{2}\right)^2 \left(\because \ \omega=\frac{2\pi}{T}\right)$$

$$=9{\cdot}87\times10^{-5}\text{ J. Ans.}$$

$$\tfrac{1}{4}\,mv^2{}_{max}=T_{max}=E \qquad \left[\because \ U+T=E \text{ (a constant)} \quad \therefore \ T_{max}=E\right]$$

or $$v_{max}=\sqrt{\frac{2E}{m}}\,\sqrt{\frac{2\times9{\cdot}87\times10^{-5}}{50\times10^{-3}}}={\cdot}063\text{ m/s}=6{\cdot}3\text{ cm/s. Ans.}$$

*5. Find the period of oscillation of a mathematical pendulum constructed by a light rigid rod carrying two masses m_1 and m_2 at distance l_1 and l_2 from the point of suspension.



Sol. Consider a small angular displacement in the anticlockwise direction.

τ (torque) $= -m_1gl_1 \sin\theta - m_2gl_2 \sin\theta$.

Applying the formula,

torque = moment of inertia × angular acceleration

$$(-m_1gl_1+m_2gl_2)\sin\theta=(m_1l_1^2+m_2l_2^2)\times\alpha$$

or $$\alpha=-\frac{(m_1l_1+m_2l_2)g}{m_1l_1^2+m_2l_2^2}\times\theta$$

$(\because \ \sin\theta=\theta$ when θ small)

$$\therefore \quad \alpha \propto -\theta.$$

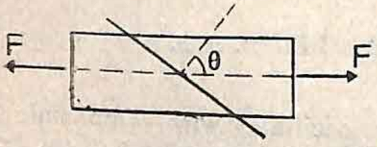
Hence motion is simple harmonic.

$$\therefore \quad \omega^2=\frac{(m_1l_1+m_2l_2)g}{m_1l_1+m_2l_2^2}$$

$$\therefore \quad T=2\pi\sqrt{\frac{(m_1l_1^2+m_2l_2^2)}{(m_1l_1+m_2l_2)g}}.$$

Elasticity

1. A bar or cross-section A is subjected to equal and opposite tensile forces F at its ends. Consider a plane section of the bar whose normal makes an angle θ with the axis of the bar. (a) What is the tensile stress on this plane ? (b) What is the shearing stress at this plane ? (c) For what value of θ is the tensile stress a maximum ? (d) For what value of θ is the shearing stress a maximum ?



Sol. The resolved part of F along the normal is the tensile force at this plane and the resolved part parallel to the plane is the shearing force at this plane.

$$\therefore \quad \text{Tensile stress} = \frac{\text{force}}{\text{area}} = \frac{F\cos\theta}{A\sec\theta} = \frac{F}{A}\cos^2\theta. \textbf{ Ans.}$$

$(\because$ area of section $= A\sec\theta)$

$$\text{(b) Shearing stress} = \frac{\text{force}}{\text{area}} = \frac{F\sin\theta}{A\sec\theta} = \frac{F}{2A}\sin 2\theta. \textbf{ Ans.}$$

(c) Obviously tensile stress at the plane is maximum when $\cos^2\theta$ is maximum, that is, $\cos\theta = 1$ or $\theta = 0^\circ$. **Ans.**

(d) Obviously shearing stress is maximum when $\sin 2\theta$ is maximum, that is, $\sin 2\theta = 1$ or $2\theta = 90^\circ$ or $\theta = 45^\circ$. **Ans.**

2. A uniform plank of Young's modulus Y is moved over a smooth horizontal surface by a constant horizontal force F. The area of transverse section of the plank is A. Find the compressive strain on the plank in the direction of the force.

Sol. Since the force at the other end is zero, the average stretching force is $\frac{F+0}{2} = \frac{F}{2}$

$$\therefore \quad \text{stress} = \frac{F}{2A}, \quad \text{strain} = \epsilon \text{ (say)}$$

Then $$Y = \frac{\frac{F}{2A}}{\epsilon} \quad \text{or} \quad \epsilon \text{ (strain)} = \tfrac{1}{2}. \frac{F}{AY}. \textbf{ Ans.}$$

3. A copper rod of length l is suspended from the ceiling by one of its ends. Find

(a) the elongation $\triangle l$ of the rod due to its own weight;

(b) the volume strain produced in terms of longitudinal strain and Poisson's ratio.

Sol. $$\text{Stress} = \frac{\text{weight}}{\text{area}} = \frac{(Al\rho)g}{A} = l\rho g.$$

Strain $= \frac{\triangle l}{l/2}$ (It is $l/2$ because the weight is effective at its centre)

$$= \frac{2\triangle l}{l}$$

$$\therefore \quad Y = \frac{l\rho g}{\frac{2\triangle l}{l}} \quad \text{or} \quad \triangle l = \tfrac{1}{2}\rho g l^2/Y. \textbf{ Ans.}$$

(b) σ (Poisson's ratio) $=\dfrac{\dfrac{\triangle D}{D}}{\dfrac{\triangle l}{l}}$ where $\triangle D$ is the change in diameter

$$\therefore \quad \frac{\triangle D}{D}=\sigma\frac{\triangle l}{l}.$$

Now $A=\frac{\pi}{4}D^2$. Differentiating, $\triangle A=\frac{\pi}{4}.\ 2D\triangle D$.

$$\frac{\triangle A}{A}=\frac{\frac{\pi}{4}\ 2D\triangle D}{\frac{\pi}{4}\ D^2}=2\ \frac{\triangle D}{D}=2\sigma\frac{\triangle l}{l}.$$

Now volume strain $=\dfrac{\triangle V}{V}=\dfrac{(A-\triangle A)(l+\triangle l)-Al}{Al}$

$=\dfrac{Al-l\triangle A+A\triangle l-Al}{Al}$ (neglecting the product $\triangle A.\triangle l$)

or volume strain $=\dfrac{A\triangle l-l\triangle A}{Al}$

$$=\frac{\triangle l}{l}-\frac{\triangle A}{A}=\frac{\triangle l}{l}-2\sigma\frac{\triangle l}{l}=(1-2\sigma)\ \frac{\triangle l}{l}.\ \textbf{Ans.}$$

*4. A sphere of radius 10 cm and mass 25 kg is attached to the lower end of a steel wire which is suspended from the ceiling of a room. The point of support is 521 cm above the floor. When the sphere is set swinging as a simple pendulum, its lowest point just grazes the floor. Calculate the velocity of the ball at its lowest position.

Young's modulus of steel $=20\times10^{10}$ N/m^2
Unstretched length of the wire $=500$ cm
Radius of the steel wire $=0{\cdot}05$ cm. **(IIT 1972)**

Sol. A wire can be considered as a spring of suitable force constant. For a wire we have

$$Y=\frac{\dfrac{F}{A}}{\dfrac{\triangle l}{l}}=\frac{Fl}{A\triangle l}$$

or
$$F=Y\frac{Al}{l}\ \triangle l$$

Now, $F=k.\Delta l$; $\therefore$ k (force constant) $=\dfrac{F}{\Delta l}=\dfrac{YA}{l}$

$$=\frac{20\times10^{10}\times\pi(\cdot05\times10^{-2})^2}{5}=\pi\times10^4\ \text{N/m}^2.$$

Now we may treat the problem as if the mass is suspended by a spring of force constant $k=10^4\pi$ N/m². Considering 'dynamics of circular motion' at the lowest point,

$$T-mg=\frac{mv^2}{r}.$$

Obviously at the lowest point the wire elongates by 1 cm.

$$\therefore\quad T=k\Delta l=\pi\times10^4\times\cdot01=\pi\times10^2\ \text{N}.$$

$$\therefore\quad \pi\times10^2-25\times9\cdot8=\frac{25\times v^2}{5\cdot21}$$

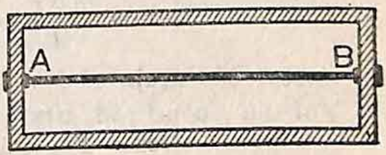
or $$v^2=\frac{(100\pi-245)\times5\cdot21}{25}=14\cdot42$$

or $$v=3\cdot8\ \text{m/s. Ans.}$$

5. An iron wire AB of length 3 m at 0°C is stretched between the opposite walls of a brass casing at 0°C. The diameter of the wire is 0·6 mm. What extra tension will be set up in the wire when the temperature of the system is raised to 40°C? Given $\alpha_{brass}=18\times10^{-6}\text{K}^{-1}$

$\alpha_{iron}=12\times10^{-6}\text{K}^{-1}$

$Y_{iron}=21\times10^{10}\ \text{N/m}^2$.



Sol. Increase in length of the iron wire $=l\alpha\Delta t=l\times12\times10^{-6}\times40$
$=48l\times10^{-5}$ m.

Increase in length of the brass tube $=l\alpha\,\Delta t=l\times18\times10^{-6}\times40$
$=72l\times10^{-5}$ m.

Since the increase in length of the brass tube is greater than that of the iron wire, the latter will be tightened. Let T be the tension in wire then

$$\text{stress}=\frac{T}{\pi r^2}=\frac{T}{\pi(3\times10^{-4})^2}=\frac{T}{9\pi\times10^{-8}}$$

$$\text{strain}=\frac{(72l-48l)\times10^{-5}}{l}=24\times10^{-5}$$

Since $$Y=\frac{\text{stress}}{\text{strain}},\ 21\times10^{10}=\frac{\dfrac{T}{9\pi\times10^{-8}}}{20\times10^{-5}}$$

or $$T=21\times10^{10}\times24\times10^{-5}\times9\pi\times10^{-8}$$
$$=(21\times24\times9\pi)\times10^{-3}$$
or $$T=14{\cdot}2\text{ N. Ans.}$$

*6. A composite rod is made by joining a copper rod, end to end, with a second rod of different material but of the same cross-section. At 25°C, the composite rod is 1 m long of which the copper rod is 30 cm. At 125°C the length of the composite rod increases by 1·91 mm. When the composite rod is prevented from expanding by holding it between two rigid walls, it is found that the constituent rods have remained unchanged in lengths in spite of rise of temperature. Find the Young's modulus and the coefficient of linear expansion of the second rod. (Y of copper$=1{\cdot}3\times10^{10}$ N/m^2 and α of copper$=17\times10^{-6}$K^{-1}.

Sol. Since total expansion = expansion of the constituent rods,

$$\therefore\quad 1{\cdot}91\times10^{-3}={\cdot}3\times17\times10^{-6}\times(125-25)+{\cdot}7\times\alpha\times(125-25)$$
$$1{\cdot}91\times10^{-3}=5{\cdot}1\times10^{-4}+70\alpha$$
or $$70\alpha=(1{\cdot}91-{\cdot}51)\times10^{-3}=1{\cdot}4\times10^{-3}$$
$$\alpha=2\times10^{-5}\text{K}^{-1}.$$

When prevented from expanding, thermal expansion is accompanied by elastic contraction. Since the lengths remain unchanged, the thermal expansion is equal to elastic contraction.

$$\therefore\quad \text{strain}=\frac{l\alpha\triangle t}{l}=\alpha\triangle t=17\times10^{-6}\times100=17\times10^{-4}$$

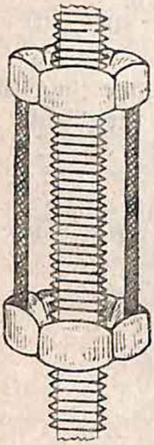
$$\text{stress}=Y\times\text{strain}=1{\cdot}3\times10^{10}\times17\times10^{-4}=22{\cdot}1\times10^{6}.$$

For the second rod, strain $=\alpha\triangle t=2\times10^{-5}\times100=2\times10^{-3}$
$$\text{stress}=Y\times\text{strain}=Y\times2\times10^{-3}.$$

But *the same stress is effective* throughout the composite rod.

$$\therefore\quad Y\times2\times10^{-3}=22{\cdot}1\times10^{6}$$
$$Y=11{\cdot}1\times10^{9}\text{ N/m}^2.\text{ Ans.}$$

7. A steel bolt is inserted into a copper tube as shown. Find the forces induced in the bolt and in the tube when the nut is turned through one revolution. Assume that length of the tube is l, the pitch of the bolt thread is h and the cross-sectional areas of the steel bolt and the copper tube are A_s and A_c respectively.



Sol. When a nut is turned, the tube is compressed and so its length decreases and the bolt is subjected to tensile force and so its length increases. The sum of these two lengths is the distance through which the nut moves.

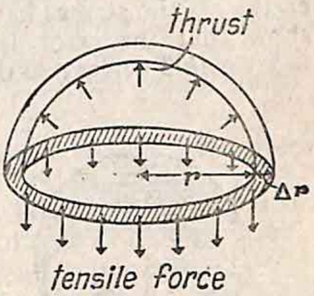
Now $$Y_s=\frac{F/A_s}{\frac{\triangle l}{l}}\quad\text{or}\quad \triangle l=\frac{F\times l}{Y_sA_s}.$$

Similarly, for the copper tube, $\triangle l = \dfrac{Fl}{Y_c A_c}$.

$$\therefore \quad h = \frac{Fl}{Y_s A_s} + \frac{Fl}{Y_c A_c}$$

or
$$F = \frac{h}{l}\,\frac{Y_s Y_c A_s A_c}{Y_s A_s + Y_c A_c}.\ \text{Ans.}$$

*8. Calculate the internal pressure that can be sustained by (a) glass tube, (b) a glass spherical flask, if in both cases the wall thickness is equal to $\triangle r = 1$ mm and the radius of the tube and flask equals $r = 2{\cdot}5$ cm. Tensile strength (σ) of glass $= 5 \times 10^7$ N/m^2.



Sol. Let us divide the sphere of glass into two hemispheres and consider equilibrium of one half.

Let σ be the tensile stress and p be the pressure on the inner wall.

The resultant of distributed tensile forces
$$= (2\pi\, r \triangle r)\sigma$$

The resultant of distributed thrusts on the wall due to pressure
$$= \pi r^2 p.$$

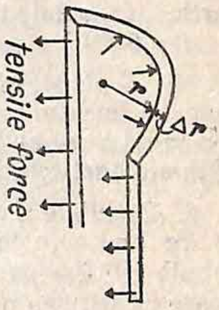
For equililibrium we have
$$\pi r^2 p = 2\pi r\ \triangle r \sigma$$

or
$$p = 2\frac{\triangle r}{r}\ \sigma.$$

If σ_b is the tensile strength, that is, breaking stress, then the maximum pressure that can be sustained by a spherical vessel is

$$p_{max} = 2\ \frac{\triangle r}{r}\,\sigma_b = 2 \times \frac{1 \times 10^{-3}}{2{\cdot}5 \times 10^{-2}} \times 5 \times 10^7$$

$= 4 \times 10^6$ Nm^{-2} $= 40$ atm (1 atm $= 10^5$ Nm^{-2}).



Similar considerations for a tube give tensile force $= (2\triangle rl)\sigma$ where l is the length of the tube.

Thrust $= (2rl)p$.

For equilibrium
$$2\triangle rl\sigma = 2rlp$$

or
$$p = \frac{\triangle r}{r}\sigma$$

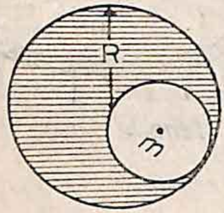
$$\therefore \quad p_{max}=\frac{\Delta r}{r}\sigma_b=\frac{10^{-3}}{2{\cdot}5\times10^{-2}}\times5\times10^{7}=2\times10^{6}\ \text{Nm}^{-2}$$

$$=20\ \text{atm. Ans.}$$

EXERCISES

Gravitation

1. Masses, assumed to be equal to m each, hang from strings of different lengths from the ends of a balance at the surface of the earth. If the strings have negligible mass and differ in length by h, show that the error in weighing, $W'-W$, is given by $W'-W=8\pi G\rho mh/3$ where ρ is the density of the earth.



2. A spherical hollow is made in a lead sphere of radius R, such that its surface touches the outside surface of the lead sphere and passes through its centre. The mass of the sphere before hollowing was M. What force does the hollowed sphere exert on a point mass **m** placed at the centre of the hollow ?

3. A train of mass 10^5 kg is at a lattitude 60°. How much will the weight of the train differ from its weight at the poles when the train (Radius of the earth = 6400 km) is

(i) stationary

(ii) travelling due east at a velocity 108 kph

(iii) travelling due west at a velocity of 108 kph ?

4. A satellite of mass 2×10^3 kg has to be shifted from the orbit equal to $2R$ to another of radius $3R$ where R is the radius of the earth. Calculate the minimum energy required.
$R=6400$ km and $g=9{\cdot}8$ m/s^2.

*5. A planet of mass m moves along an ellipse around the sun so that its maximum and minimum distances from the sun (mass $=m_s$) are equal to r_1 and r_2 respectively. Find the angular momentum L of this planet relative to the centre of the sun.

6. Calculate the force of interaction between a lead ball of mass 10 kg and volume 800 cc placed at a distance 1 m from the centre a bubble of the same volume when both are situated in water. Is it a case of attraction or repulsion ?

7. A mine pierces the earth along one of its diameters. In what time will a body released into the mine reach the centre of the earth ? $R=6400$ km and $g=10$ m/s^2.

8. (a) What is the total energy of an artificial satellite circling round the earth in an orbit of radius R ?

(b) What is the escape velocity for this satellite ?

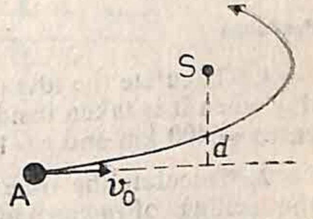
(c) Show that the satellite obeys Kepler's third law, that is, the ratio the cube of its orbital radius to square of its time of revolution is a constant. (Roorke 1981)

***9.** Inside a uniform sphere of density ρ there is a spherical cavity whose centre is at a distance l from the centre of the sphere. Find the strength of the gravitational field inside the cavity.

10. A uniform sphere has a mass M and radius R. Find the pressure inside the sphere caused by attraction of the layers towards the centre of the sphere as a fuction of distance r from its centre.

11. A double star is a system of two stars moving around the centre of mass of the system due to gravitation. Find the distance between the components of the double star if the total mass equals M and the period of revolution is T.

12. A planet of mass m moves around the sun (mass M) along an elliptical path so that its minimum distance from the sun is r_1 and the maximum distance is r_2. Using Kepler's laws, that is, period of revolution is the same, find its period of revolution around the sun.



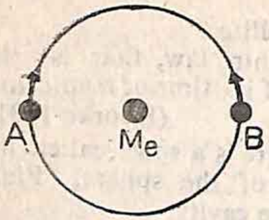
***13** A cosmic ray body A moves to the sun S with velocity v_0 when far from the sun and aiming along a line having perpendicular distance d from the sun. Find the minimum distance of this body from the sun. Take M as the mass of the sun.

***14.** Find the third cosmic velocity v_3, that is, the minimun velocity relative to the earth's surface to drive out a body out of the solar system in terms of the orbital speed v_0 of a body round the earth and the orbital speed V_0 of a body round the sun. The rotation of the earth about its own axis is to be neglected.

15. Calculate minimum work that has to be performed to carry a spaceship of mass $m=2\times10^3$ kg from the surface of the earth to the moon. Mass of the earth$=6{\cdot}0\times10^{24}$ kg, mass of the moon $=7{\cdot}3\times10^{22}$, radius of the earth$=6{\cdot}4\times10^6$ m and radius of the moon$=1{\cdot}7\times10^6$ m.

16. A spaceship is launched into a circular orbit close to the earth's surface. What additional velocity has to be imparted to the spaceship to overcome the earth's gravitational pull ? $g=9{\cdot}8$ m/s^2 and $R=6400$ km.

17. A cosmic body is in equilibrium on the line joining the sun to the earth. How far is the body from the earth ? Take m, M and d as the mass of the earth, mass of the sun and distance between the centres of the sun and the earth.



18. Consider two satellites A and B of equal mass, moving in the same circular orbit of radius r around the earth but in opposite senses of rotation and therefore on collision course.

(a) Find the total mechanical energy E_A+E_B of the two-satellite-plus-earth system before collisions.

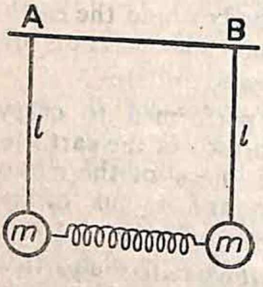
(b) If the collision is completely inelastic, find the total mechanical energy immediately after collision. Describe the subsequent motion of the combined satellite.

19. A particle of mass m is subject to an attractive central force of magnitude k/r^2, k being a constant. If at the instant when the particle is at an extreme position in its closed path, at a distance a from the centre of the force, its speed is $\sqrt{\frac{k}{2ma}}$, find the other extreme position.

Pendulum

1. Calculate the loss or gain of a correct pendulum clock in a day when it is taken inside a mine of depth 200 m. Radius of the earth = 6400 km and $g=10\text{ m/s}^2$.

2. Calculate the time period of a simple pendulum hanging from the ceiling of a car negotiating round a curved path of radius R at speed v.



3. Two simple pendulums each of length l are connected by a weightless spring as shown in the figure. In the equilibrium position the spring is not deformed. Find the frequencies of small oscillations when

(a) they are deflected in one plane through the same angles in one direction,

(b) in opposite directions.

4. A small bob of mass 50 gm oscillates as simple pendulum, with amplitude 5 cm and period 2 s. Find the velocity of the bob and the tension in the supporting thread when the velocity of the bob is maximum.

5. In a ballon expedition, seconds pendulum carried on the balloon, on reaching the topmost position is found to loss 400 seconds a day. Estimate the height of ascent, assuming the earth's radius to be 6460 km.

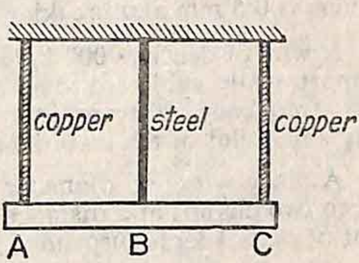
(JEE, WB 1979)

Elasticity

1. A steel wire of cross-sectional area 0·5 mm^2 is held between fixed supports. If the tension in the wire is negligible and it is just taut at a temperature of 20°C, determine the tension when the temperature falls to 0°C. Young's modulus of elasticity is 21×10^{10}N/m^2 and the coefficient of linear expansion is 12×10^{-6}K^{-1}. Assume that distance between the supports remains the same. **(IIT 1973)**

2. A wire 3 metres in length and 1 mm in diameter at 30°C, is kept in a low temperature chamber at -170°C and is stretched by hanging a 10 kg mass at one end. Calculate the change in length of the wire. Coefficient of linear expansion $= 12 \times 10^{-6}$K^{-1} and Young's modulus $= 1{\cdot}995 \times 10^{11}$N/m^2. **(IIT 1968)**

3. A heavy plank of mass 100 kg hangs on three vertical wires of equal length arranged symmetrically (see figure). Find the tensions in the wires if the middle wire is of steel and the other two are of copper. All the wires have the same cross-section. Consider the modulus of elasticity of steel to be double that of copper.



4. A copper plate of length 1 m is rivetted to two steel plates of the same length and section at 0°C. Calculate the tensions in copper and steel plates when heated to 20°C. Assume that

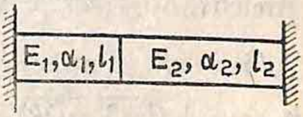
$$Y_{copper} = \tfrac{1}{2} Y_{steel} \text{ and } Y_{steel} = 2 \times 10^{10}\text{N/m}^2.$$

$\alpha_c = 18 \times 10^{-6}$K^{-1}; $\alpha_s = 11 \times 10^{-6}$K^{-1} and area of each plate $= 50$ cm^2.

5. A wire of length $2l$ is stretched between two supports. A heavy weight of mass M is suspended exactly from the middle of the wire. The cross-sectional area is A and its modulus of elasticity is E. Determine the distance by which mid point is lowered under the weight, considering it small.

6. Find the maximum permissible linear velocity of a rotating thin lead ring (axis of rotation is the axis of the ring) if the tensile strength of lead is $\sigma = 2 \times 10^7$ N/m^2 and its density $= 11300$ kg/m^3.

7. Two rods made of different materials are placed between massive walls (see figure). The cross-section of the rods is A and their respective lengths are l_1 and l_2. The rods are heated by t°C. Find the force with which the rods act on each other if their coefficients of expansion are α_1 and α_2



and their Young's modulii are E_1 and E_2. Neglect the deformation of the walls.

8. Find the elastic deformation energy per unit volume in fresh water at a depth of $h=1$ km. Given that β (compressibility of water) $=5\times10^{-10}$ units.

9. Find the elastic deformation energy of a steel rod of mass $m=5$ kg when stretched to a tensile strain $\epsilon=10^{-3}$ m. Density of steel $(\rho)=7800$ kg/m^3 and E (Young's modulus)$=2\times10^{11}$ N/m^2.

10. Calculate the maximum pressure that a spherical glass boiler of radius 10 cm and wall-thickness 2 mm can bear if the breaking stress of glass is 5×10^7 Pa.

11. Calculate the maximum pressure that a cylindrical boiler of copper can withstand if the radius of the boiler is 5 cm and its wall-thickness is 0·5 mm and breaking strength of copper 3×10^8 Pa.

12. A wire of density 9000 kg/m^3 is stretched between two clamps 1 m apart, while subjected to an extension of 0·05 cm. What is the lowest frequency of transverse vibrations in the wire, assuming Young's modulus of the material of the wire 9×10^{10} N/m^2. (IIT '75)

13. A steel wire of diameter 1 mm is stretched horizontally between two clamps at a distance 2 m apart from each other. A weight of mass $\frac{1}{4}$ kg is suspended from the midpoint O of the wire. What will be the resulting descent of the point O? E (Young's modulus of steel)$=2\times10^{11}$ N/m^2.

14. A 6 kg weight is fastened to the end of a steel wire of unstretched length 60 cm. It is whirled in a vertical circle and has an angular velocity of 2 rev/sec at the bottom of the circle. Area of cross-section of the wire is 0·05 cm^2. Calculate the elongation of the wire when the weight is at the lowest point of the path. Young's modulus of elasticity of steel$=2\times10^{11}$ Pa.

15. A disc of mass m is attached to a spring of unstretched length l_0 and force constant k. If the disc is rotated in a smooth horizontal plane, show that the radius of the uniform circular motion is $\dfrac{kl_0}{k-4\pi^2mf^2}$ and tension in the spring is $\dfrac{4\pi^2mkl_0f^2}{k-4\pi^2mf^2}$ where f is the frequency of revolution.

16. A rubber cord of unstretched length l is attached to a ball of mass m. The cord is fixed to a point and the ball is taken to the horizontal unstretched position and released. Show that the cord stretches by $\triangle l=\dfrac{3mg}{k}$ at the lowest position of the ball provided $\triangle l$ is small. Further show that the speed of the ball at the bottom is $v=\sqrt{2g(l-3mg/2k)}$.

TEST YOURSELF

1. Give reasons of the following : $3 \times 10 = 30$

(i) A simple pendulum is carried in a balloon above the surface of the earth. Will it loose or gain ?

(ii) Can you find the position of the centre of mass of a body by arranging a simple pendulum with it in your laboratory ? In a space laboratory ?

(iii) Two pendula, each consisting of a disc attached to light bar, are identical except for the coupling between disc and bar. In one the bar is rigidly mounted to the disc; in the other ball-bearings are used so that the disc would be free to spin about the end of the bar. Both pendula are hung, pulled aside to the same height, and released. Which has the greater period ? Explain.

(iv) Inside a spaceship what difficulties would you encounter ? In walking ? In jumping ? In drinking ?

(v) Because the earth bulges near the equator, the source of Missisipi River, although, high above sea level, is nearer to the centre of the earth than is its mouth. How can river flow 'uphill' ?

(vi) When a force is applied at an inclination to a face of a fixed rigid cube, does it produce the same tensile and shearing stress at that face ? If not, explain why ?

(vii) A liquid surface is horizontal because it has no definite shape. Is it true or false ?

(viii) Why does the earth impart to all bodies the same acceleration irrespective of their mass ? Is it true for other forces, pull of engine on train for example ?

(ix) In what conditions will bodies inside a spaceship be 'weightless' ?

(x) When a rod clamped at one end is heated is it strained ? When clamped at both ends ?

2. Answer the following objectives : $2 \times 10 = 20$

(i) A force F is applied in a vertical plane perpendicular to the top face of parallelopiped block of length, breadth and height a, b and c respectively at inclination θ to the vertical. The lower face of the block containing length and breadth is fixed to a horizontal surface. The shearing stress at the top face is

(a) $\dfrac{F\cos\theta}{bc}$, (b) $\dfrac{F\sin\theta}{ab}$, (c) $\dfrac{F\cos\theta}{ca}$. (d) $\dfrac{F\sin\theta}{ca}$.

(ii) In the above example the shearing stress on the vertical faces containing b and c sides is

(a) zero, (b) $\dfrac{F\cos\theta}{bc}$, (c) $\dfrac{F\sin\theta}{bc}$. (d) $\dfrac{F\cos\theta}{ab}$.

(iii) In the above examples the tensile stress on the top face is

(a) $\frac{F\sin\theta}{ab}$, (b) $\frac{F\cos\theta}{ca}$, (c) $\frac{F\sin\theta}{bc}$, (d) $\frac{F\cos\theta}{ab}$.

(iv) If T is the period of a planet and R is the radius of its orbit then (a) $T \propto R$, (b) $T^2 \propto R^3$, (c) $T \propto R^3$, (d) $T \propto \frac{1}{R}$.

(v) The compressibility of water is 0·5 GPa^{-1}, its bulk modulus is (a) $\cdot5 \times 10^9$N/m^2, (b) $\cdot5 \times 10^{-9}$N/m^2, (c) 2×10^9N/m^2, (d) 2×10^{-9}N/m^2.

(vi) A wire of length 1 m is stretched without tenstion at 30°C between two fixed supports. If the temperature falls to 0°C, the strain produced in the wire ($\alpha = 12 \times 10^{-6}$K^{-1}) is

(a) 4×10^{-6}, (b) 4×10^{-6} m, (c) 36×10^{-5} m, (d) 36×10^{-5}.

(vii) If 3×10^{-3} J of work is done in stretching a wire through 1 mm, the work done in stretching it through 3 mm (a) 9×10^{-3} J, (b) 1×10^{-3} J, (c) 27×10^{-3} J, (d) cannot be said unless force is given.

(viii) A spaceship approches the moon whose mass and radius are M and R along a line which is almost tangential to the moon's surface. At the moment of maximum approach the brake rocket was fired for a short while and the spaceship was transferred into a satellite of the moon. The change in velocity of the spaceship is

(a) $\sqrt{\frac{GM}{R}}\sqrt{(2-1)}$, (b) $\sqrt{\frac{GM}{R}}$, (c) $\sqrt{\frac{2GM}{R}}$, (d) $\sqrt{\frac{GM}{2R}}$.

(ix) The gravitational field inside a spherical shell is

(a) zero, (b) $-\frac{GM}{a}$, (c) $-\frac{GM}{a^2}$, (d) infinite.

(x) A satellite of mass m, initially at rest on the earth, is launched in circular orbit at a height equal to the radius of the earth. The minimum energy required is (a) $\frac{3}{4}gR$, (b) $\frac{1}{2}gR$, (c) $\frac{1}{4}gR$, (d) 3/4 gR.

Answer any five of the following : $5 \times 10 = 50$

3. (a) An artificial satellite revolves 15 times a day around the earth in circular orbit of radius 7000 km. Find the gravitational intensity at the site of the satellite.

(b) A satellite of mass 250 kg revolves around a planet of mass 5×10^{30} kg in a circular orbit of radius $6{\cdot}67 \times 10^6$ m. Find the binding energy of the planet+satellite system.

$G = 6{\cdot}67 \times 10^{-11}$ Nm^2 kg^{-2}.

4. (a) A steel rod is subjected to 10 kgf tensile force at 27°C. Calculate the temperature at which it is freed from strain if the area of cross-section of the rod is 5 mm^2 and α of steel $= 10 \times 10^{-6}$K^{-1} and Y of steel $= 2 \times 10^{11}$N/m^2 and $g = 10$ m/s^2.

(b) The radius of the wooden wheel of a bullock cart is 75 m. A steel ring of thickness 1 cm is put on it by heating to 200°C. Calculate the force per unit area with which the ring grips over wheel when it is cooled to 30°C. Y of steel $= 200$ GPa and α of steel $= 10^{\;6}\mathrm{K}^{-1}$.

5. Two point masses, each equal to M, are placed at a distance $2a$ apart. Show that a small mass m placed midway between them on the line joining them will be equilibrium and if it is slightly displaced a little from this position along the line perpendicular to the line joining the masses it will execute simple harmonic motion. Calculate the frequency of these oscillations.

6. A satellite revolving in a circular equatorial orbit of radius $R = 2 \times 10^4$ km from east to west. Calculate the interval after which it will appear at the same equatorial town. Given that radius of the earth $= 6400$ km and g (acceleration due to gravity) $= 10\ \mathrm{m/s^2}$.

7. (a) Calculate the time period of a pendulum constructed by a light rigid rod carrying three bobs of masses m_1, m_2 and m_3 at distances l_1, l_2 and l_3 from the point of suspension.

(b) Calculate the time period of oscillation of a seconds pendulum in an elevator moving up with constant acceleration $2\ \mathrm{m/s^2}$. Take $g = 10\ \mathrm{m/s^2}$.

8. A particle of mass m is subjected to an attractive central force of magnitude k/r^2, k being a constant. If at the instant when the particle is at an extreme position in its orbit, at a distance 'a' from the centre of force, its speed is $\sqrt{k/2ma}$, find the other extreme position.

Answers to Exercises

Gravitation

2. $F = \dfrac{GMm}{2R^2}$. 3. (i) 86·3 kg, (ii) 108·5 kg, (iii) 64·0 kg.

4. $1{\cdot}04 \times 10^{10}$ joule. 5. $L = m\sqrt{2Gm_s \dfrac{r_1 r_2}{r_1 + r_2}}$.

6. $-4{\cdot}8 \times 10^{-10}$ N. 7. 20·9 min. 8. (a) $-\frac{1}{2}\dfrac{GMm}{R}$.

9. $\frac{4\pi}{3} G\rho l$. 10. $p = \dfrac{3GM^2}{8\pi R^4}\left(1 - \dfrac{r}{R^2}\right)$.

11. $d = \left(\dfrac{GMT^2}{4\pi^2}\right)^{\frac{1}{3}}$. 12. $T = 2\pi\sqrt{\dfrac{(r_1 + r_2)^3}{8GM}}$.

13. $d_{min}=\frac{GM}{v_0^2}\left(\sqrt{1+\frac{d^2v_0^4}{G^2M^2}}-1\right)$.

14. $v_3=\sqrt{2v_0^2+(\sqrt{2}-1)^2V_0^2}$. **15.** $1{\cdot}12\times10^{11}$ joule.

16. $3{\cdot}279\times10^3$ m/s. **17.** $\frac{\sqrt{m}}{\sqrt{m}+\sqrt{M}}d$.

18. $-\frac{GM_em}{r}$; $-\frac{2GM_em}{r}$ combined body falls directly down.

19. $\frac{a}{3}$.

Pendulum

1. 1·35 s. **2.** $2\pi\left(\frac{l^2}{g^2+\frac{v^4}{R^2}}\right)^{\frac{1}{4}}$. **3.** $2\pi\sqrt{\frac{l}{g}}$;

$f=\frac{1}{2\pi}\sqrt{\frac{g}{l}+\frac{2k}{m}}$. **4.** ·16 m/s; ·49 newton. **5.** 30 km.

Elasticity

1. 25·2 N. **2.** ·0053 m. **3.** $T_{copper}=25$ kgf; $T_{steel}=50$ kgf.

4. 56×10^3 N. **5.** $l\sqrt[3]{\frac{Mg}{EA}}$. **6.** 42 m/s.

7. $F=At.\frac{\alpha_1l_1+\alpha_2l_2}{\frac{l_1}{E_1}+\frac{l_2}{E_2}}$. **8.** $2{\cdot}4\times10^4$ Jm^{-3}. **9.** 64 J.

10. 2×10^6 Pa. **11.** 3×10^6 Pa. **12.** 35 Hz.
13. $2{\cdot}5\times10^{-2}$ m. **14.** $3{\cdot}36\times10^{-6}$ m.

Answers of Test Questions

2. (i) (b), (ii) (b), (iii) (d), (iv) (b), (v) (c), (vi) (d), (vii) (c), (viii) (a), (ix) (a), (x) (a).
3. 7·7 m/s^2; $-6{\cdot}25\times10^{15}$ joule. **4.** 17·2°C; $1{\cdot}44\times10^6$ N/m^2.

5. $f=\frac{1}{2\pi}\sqrt{\frac{GM}{a^3}}$. **6.** 5 h 48 min.

7. $T=2\pi\sqrt{\dfrac{m_1l_1^2+m_2l_2^2+m_3l_3^2}{(m_1l_1+m_2l_2+m_3l_3)g}}$; 1·82s. **8.** $\dfrac{a}{3}$.

Solution to Exercises

Gravitation

1. $$W'-W=\frac{Gm\frac{4\pi}{3}R^3\rho}{(R+h)^2}-\frac{Gm\frac{4\pi}{3}R^3\rho}{R^2}$$

$$=\frac{4\pi}{3}Gm\rho\left[R\left(1+\frac{h}{R}\right)^{2}-R\right]$$

$$=\frac{4\pi}{3}Gm\rho\left[R\left(1+\frac{2h}{R}\right)-R\right]\quad(\because\ h<<R)$$

$$=\frac{8\pi}{3}Gm\rho h.$$ **Proved.**

2. The gravitational force = gravitational field × mass.

By the principle of superposition of fields,

$$g=g_1+g_2$$

where g = gravitational field at the site of m due to entire mass

g_1 = gravitational field at the same point due to hollowed sphere

g_2 = gravitational potential at the same point due to mass in the hollowed portion.

Now g = field at a point inside a solid sphere at a distance $\frac{R}{2}$

$$=\frac{GM}{R^3}\cdot\frac{R}{2}\text{ (taking only the absolute value)}=\frac{GM}{2R^2}$$

g_2 = field at the centre of a solid sphere of radius $\frac{R}{2}=0$

$$\therefore\ \frac{GM}{2R^2}=g_1+0\quad\text{or}\quad g_1=\frac{GM}{2R^2}.$$

$\therefore$ Force on m due to hollowed sphere $=\dfrac{GMm}{2R^2}$. **Ans.**

3. See Ex. 2 for deduction of the general expression for the apparent weight W which is

$$W = mg\left[1 - \frac{\omega^2 R\cos^2\lambda}{g} - \frac{2v\omega\cos\lambda}{g}\right]$$

At the pole $\lambda = \frac{\pi}{2}$

$\therefore \quad W_{pole} = mg$

$\therefore \quad W_{pole} - W = m\omega^2 R\cos^2\lambda + 2mv\omega\cos\lambda$

ω (angular speed of the earth) $= \frac{2\pi}{86400} = 7{\cdot}27 \times 10^{-5}$ rad/s.

(i) When stationary, $v = 0$

$$\begin{aligned}\therefore \quad \triangle W = W_{pole} - W &= m\omega^2 R\cos^2\lambda\\ &= 10^5 \times (7{\cdot}27 \times 10^{-5})^2 \times (6400 \times 10^3) \times \cos^2 60^\circ\\ &= 7{\cdot}27^2 \times 64 \times {\cdot}5^2 = 845{\cdot}6 \text{ N}\\ &= 86{\cdot}3 \text{ kg.}\end{aligned}$$ **Ans.**

(ii) Here $v = 108$ kph $= \frac{108 \times 1000}{3600} = 30$ m/s.

$$\begin{aligned}\therefore \quad \triangle W &= 845{\cdot}6 \text{ N} + 2 \times 10^5 \times 30 \times 7{\cdot}27 \times 10^{-5} \times {\cdot}5\\ &= 845{\cdot}6 + 218{\cdot}1 = 1063{\cdot}7 \text{ N} = 108{\cdot}5 \text{ kg.}\end{aligned}$$ **Ans.**

(iii) Here $v = -108$ kph $= -30$ m/s.

$\therefore \quad \triangle W = 845{\cdot}6 - 218{\cdot}1 = 627{\cdot}5 = 64{\cdot}0$ kg. **Ans.**

4. Total mechanical energy of a satellite in a circular orbit of radius $r = -\frac{1}{2}\frac{GMm}{r}$.

$\therefore \quad E_1$ (energy in the first orbit) $= -\frac{1}{2}\frac{GMm}{2R}$

E_2 (,, in the second ,,) $= -\frac{1}{2}\frac{GMm}{3R}$.

$\therefore$ Energy (minimum) required $= E_2 - E_1 = \frac{1}{2}\,\frac{GMm}{R}\left(\frac{1}{2} - \frac{1}{3}\right)$

$$= \frac{GMm}{12R}$$

$$= \frac{mgR^2}{12R} \quad (\because \ GM = gR^2)$$

$$=\frac{mgR}{12}=\frac{2\times10^3\times9{\cdot}8\times6400\times1000}{12}$$

$$=1{\cdot}04\times10^{10}\text{ joule.}\quad\textbf{Ans.}$$

5. At the minimum and maximum distant positions of the planet its velocities are at right angles to its position vectors. Remembering that moment of momentum is also angular momentum we have by conservation of angular momenta at the two positions

$$(mv_1)r_1=(mv_2)r_2$$

or $$v_1r_1=v_2r_2.$$

Considering conservation of energy

$$-\frac{Gm_sm}{r_1}+\tfrac{1}{2}mv_1^2=-\frac{Gm_sm}{r_2}+\tfrac{1}{2}mv_2^2.$$

Putting the value of v_2 in terms v_1

$$\therefore\quad -\frac{Gm_sm}{r_1}+\tfrac{1}{2}mv_1^2=-\frac{Gm_sm}{r_2}+\tfrac{1}{2}m\frac{v_1^2r_1^2}{r_2^2}$$

or $$Gm_sm\left(\frac{1}{r_2}-\frac{1}{r_1}\right)=\tfrac{1}{2}mv_1^2\left(\frac{r_1^2}{r_2^2}-1\right)$$

or $$Gm_s\frac{r_1-r_2}{r_1r_2}=\tfrac{1}{2}v_1^2\ \frac{(r_1-r_2)(r_1+r_2)}{r_2^2}\ (\because\quad r_1\neq r_2)$$

or $$v_1=\sqrt{\frac{2Gm_sr_2}{r_1(r_1+r_2)}}$$

$$L=mv_1r_1=m\sqrt{2Gm_s\frac{r\ r_2}{r_1+r_2}}.\quad\textbf{Ans.}$$

6. The lead sphere of mass 10 kg behaves like a mass equal to its 'own mass minus the mass of equal volume of water' and the bubble behaves like a mass in the same way as a mass equal to its own mass which is zero minus the mass of equal volume of mass in water. Thus bubble be haves like a negative mass in water.

$\therefore$ m_1 (equivalent mass of lead ball) $=10-800\times10\times1000$
$=10-{\cdot}8=9{\cdot}2$ kg.

$m_2=$ (equivalent mass of bubble) $=0-{\cdot}8=-{\cdot}8$ kg.

$$\therefore\quad\text{Force between the two}=\frac{G\times9{\cdot}2\times(-{\cdot}8)}{1^2}$$

$$=-6{\cdot}6\times10^{-11}\times9{\cdot}2\times{\cdot}8$$
$$=-4{\cdot}8\times10^{-10}\text{ N.}\quad\textbf{Ans.}$$

7. Let x be distance of the mass m from the centre at any instant. Remembering that the gravitational field inside a sphere at a distance x from the centre is $\frac{GM}{R^3}x$ we have,

$\therefore$ Gravitational force $= \frac{GMm}{R^3}x$ (in the $-$ve direction of x)

$$= -\frac{GMm}{R^3}x \text{ (in the +ve direction of } x)$$

where M = mass of the earth
and R = radius of the earth.

$$\therefore \quad -\frac{GMm}{R^3}x = m\frac{d^2x}{dt^2}$$

$$\therefore \quad \frac{d^2x}{dt^2} = -\frac{GM}{R^3}x = -\frac{g}{R}x \quad (\because GM = gR^2)$$

Thus $\frac{d^2x}{dt^2} \propto -x$ and hence motion is simple harmonic,

Here $\left(\frac{2\pi}{T}\right)^2 = \frac{g}{R}$

or
$$T = 2\pi\sqrt{\frac{R}{g}} = 2\pi\sqrt{\frac{6400 \times 1000}{10}} = 5027 \text{ s}$$
$$= 83{\cdot}8 \text{ min.}$$

$$\therefore \quad \frac{T}{4} = \frac{83{\cdot}8}{4} = 20{\cdot}9 \text{ min.}$$ **Ans.**

8. (a) Total energy = potential energy + kinetic energy

$$= -\frac{GMm}{R} + \frac{1}{2}mv^2.$$

By dynamics of circular motion

$$\frac{GMm}{R^2} = \frac{mv^2}{R} \quad \text{or} \quad v^2 = \frac{GM}{R}$$

$$\therefore \quad E \text{ (total energy)} = -\frac{GMm}{R} + \frac{1}{2}\frac{GMm}{R}$$

$$= -\frac{1}{2}\frac{GMm}{R}.$$ **Ans.**

(b) Let v be the velocity of escape.

Then $$\frac{1}{2}mv^2=\frac{1}{2}\ \frac{GMm}{R}$$

$$v=\sqrt{\frac{GM}{R}}.$$

Note. Note here that the escape velocity of an encircling particle is different from that of a stationary particle.

(c) Let a be the radius of the orbit.

Then $$\frac{GMm}{a^2}=m\omega^2a=m\left(\frac{2\pi}{T}\right)^2a$$

or $$\frac{T^2}{a^3}=\frac{4\pi^2m}{GMm}=\frac{4\pi^2}{GM}\text{ (a constant)}$$

$\therefore$ $$a^3\propto T^2.\quad \textbf{Proved.}$$

9. g (gravitational field at the point due to the sphere)

$$=\frac{GM}{R^3}\ x=\frac{G}{R^3}\left(\frac{4\pi}{3}R^3\rho\right)l\quad (\because\quad x=l\text{ here})$$

$$=\frac{4\pi}{3}G\rho l$$

g_1 (gravitational field at the same point due to cavity mass)$=0$.

By the 'principle of superposition of fields'

$g=g_1+g_2$ where g_2 is the required field

or $$\frac{4\pi}{3}G\rho l=0+g_2$$

or $$g_2=\frac{4\pi}{3}G\rho l.\quad \textbf{Ans.}$$

10. Consider a thin layer bounded by two concentric spherical shells of radii r and $r+dr$. Let the pressure at distances r and $r+dr$ be p and $p-dp$.

Inward gravitational pull on the layer

$$=\left(\frac{GM}{R^3}r\right)(4\pi r^2\,dr\rho)$$

(force = field × mass)

$$=\frac{4\pi GM\rho}{R^3}\,.\,r^3dr$$

$$=\frac{4\pi GM}{R^3}\left(\frac{3M}{4\pi R^3}\right)r^3dr\left(\because \quad \rho=\frac{M}{\frac{4\pi}{3}R^3}=\frac{3M}{4\pi R^3}\right)$$

$$=\frac{3GM^2}{R^6}.\ r^3dr.$$

Outward force due to press difference $= -4\pi r^2\,dp$.
For equilibrium we have

$$-4\pi r^2\,dp=\frac{3GM^2}{R^6}\ r^3dr$$

or
$$-dp=\frac{3GM^2}{4\pi R^6}\ rdr.$$

Integrating,
$$-p=\frac{3GM^2}{8\pi R^6}r^2+c \text{ (a constant)}..$$

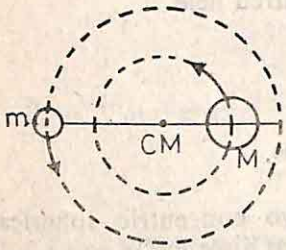
When $r=R,\quad p=0.$

$$\therefore \quad 0=\frac{3GM^2}{8\pi R^6}.\ R^2+c.$$

$$\therefore \quad -p=\frac{3GM^2}{8\pi R^6}\ r^2-\frac{3GM^2}{8\pi R^6}\ R^2$$

or
$$p=\frac{3GM^2}{8\pi R^4}\left(1-\frac{r^2}{R^2}\right).\quad \textbf{Ans.}$$

11. Let d be the distance between the centres of the sun and d_1 and d_2 be their distances from the centre of mass.



Then
$$d_1=\frac{d}{m_1+m_2}\times m_2$$

and
$$d_2=\frac{d}{m_1+m_2}\times m_1$$

$$\left(\text{from } x_{cm}=\frac{m_1x_1+m_2x_2}{m_1+m_2}\right).$$

The two stars move in circular orbits of different radii, namely d_1 and d_2, under the mutual gravitational pull.

$$\therefore \quad \frac{Gm_1m_2}{d^2}=m_1\omega^2d_1$$
$$=m_2\omega^2d_2.$$

Given that $m_1+m_2=M$

$$\therefore \quad d_1=\frac{d}{M}m_2 \quad \text{and} \quad d_2=\frac{d}{M}m_1$$

$$\therefore \quad \frac{Gm_1(M-m_1)}{d^2}=m_1\omega^2.\left[\frac{d}{M}(M-m_1)\right]$$

or
$$GM=\omega^2 d^3$$

or
$$d^3=\frac{GMT^2}{4\pi^2}$$

or
$$d=\left(\frac{GMT^2}{4\pi^2}\right)^{\frac{1}{3}}. \quad \textbf{Ans.}$$

12. Since the period of revolution is the same by Kepler's laws it is sufficient to consider the motion along the circle whose radius is equal to the mean of the major and minor axes of the given ellipse.

$$\therefore \quad \text{radius of the circular orbit}=\frac{r_1+r_2}{2}=r \text{ (say)}$$

$$\therefore \quad \frac{GMm}{r^2}=m\omega^2 r$$

or
$$\omega^2=\frac{GM}{r^3} \quad \text{or} \quad T=2n\sqrt{\frac{r^3}{GM}}$$

or
$$T=2\pi\sqrt{\frac{(r_1+r_2)^3}{8GM}}. \quad \textbf{Ans.}$$

13. By energy conservation we have

$$\frac{1}{2}mv_0{}^2=\frac{1}{2}mv^2-\frac{GMm}{d_{min}} \quad \text{..} \quad \text{(i).}$$

By 'conservation of angular momentum of the body about S' (remember moment of momentum is the angular momentum)

$$mv_0 d=mvd_{min} \quad \text{..} \quad \text{(ii).}$$

Eliminating v between (i) and (ii)

$$\frac{1}{2}v_0{}^2=\frac{1}{2}\left(\frac{dv_0}{d_{min}}\right)^2-\frac{GM}{d_{min}}.$$

$$v_0{}^2=d^2v_0{}^2x^2-2GMx, \qquad \text{where } x=\frac{1}{d_{min}}$$

or
$$d^2v_0{}^2x^2-2GMx-v_0{}^2=0$$

or $$x=\frac{2GM\pm\sqrt{4G^2M^2+4v_0^4d^2}}{2d^2v_0^2}$$

or $$d_{min}=\frac{d^2v_0^2}{GM+\sqrt{G^2M^2+v_0^4d^2}}$$
$$=\frac{d^2v_0^2(\sqrt{G^2M^2+v_0^4d^2}-GM)}{v_0^4d^2}$$

or $$d_{min}=\frac{GM}{v_0^2}\left(\sqrt{1+\frac{d^2v_0^4}{G^2M^2}}-1\right). \text{ Ans.}$$

14. Let d be the distance between the earth and the sun.

Then $\frac{mV_0^2}{d}=\frac{GMm}{d^2}$ or $V_0=\sqrt{\frac{GM}{d}}$, where M is the mass of the sun.

Also $\frac{mv_0^2}{R_e}=\frac{GM_e}{R_e^2}$ or $v_0=\sqrt{\frac{GM_e}{R_e}}$, where M_e is the mass of the earth and R_e is the radius of the earth.

Velocity of escape of a body from the earth's orbit

$$=\sqrt{\frac{2GM}{d}}=\sqrt{2}V_0.$$

Velocity relative to the earth $=\sqrt{2}V_0-V_0=(\sqrt{2}-1)V_0$

$$\therefore \quad \frac{1}{2}mV_3^2=\frac{GM_em}{R_e}+\frac{1}{2}m(\sqrt{2}-1)^2V_0^2$$

(here the first term is the gravitational potential energy due to the earth)

or $$V_3^2=\frac{2GM_e}{R_e}+(\sqrt{2}-1)^2V_0^2$$
$$=2v_0^2+(\sqrt{2}-1)^2V_0^2$$

or $$V_3=\sqrt{2v_0^2+(\sqrt{2}-1)^2V_0^2}. \text{ Ans.}$$

15. Consider a point at a distance x from the centre of the earth ($x>R_e$).

Force on the spaceship $=\frac{GM_em}{x^2}-\frac{GM_mm}{(d-x)^2}$, where d is the distance of centre to centre of the earth and moon.

$\therefore$ W (work done to carry it from $x=R_e$ to $x=d-R_m$)

$$=\int_{R_e}^{d-R_m}\left\{\frac{GM_em}{x^2}-\frac{GM_mm}{(d-x)^2}\right\}dx$$

$$=GM_em\left[-\frac{1}{x}\right]_{R_e}^{d-R_m}-GM_mm\left[\frac{1}{d-x}\right]_{R_e}^{d-R_m}$$

$$=GM_em\left[\frac{1}{R_e}-\frac{1}{d-R_m}\right]-GM_mm\left[\frac{1}{R_m}-\frac{1}{d-R_e}\right].$$

Taking d to be very large,

$$W\approx\frac{GM_em}{R_e}-\frac{GM_mm}{R_m}.$$

$$=Gm\left(\frac{M_e}{R_e}-\frac{M_m}{R_m}\right)$$

$$=6{\cdot}6\times10^{-11}\times2\times10^3\left(\frac{6{\cdot}0\times10^{24}}{6{\cdot}4\times10^6}-\frac{7{\cdot}3\times10^{22}}{1{\cdot}7\times10^6}\right)$$

$$=12{\cdot}6\times10^{-8}(9{\cdot}4\times10^{17}-4{\cdot}3\times10^{16})$$

or $\quad W=1{\cdot}12\times10^{11}$ Joule. **Ans.**

16. The orbital velocity round the earth $=v_0=\sqrt{\dfrac{GM}{R}}=\sqrt{gR}$

$(\because\ GM=gR^2)$.

The escape velocity from the surface of the earth

$$v_e=\sqrt{\frac{2GM}{R}}=\sqrt{2gR}.$$

$\therefore$ The additional velocity needed $=\triangle v=v_e-v_0=\sqrt{gR}(\sqrt{2}-1)$

$$=\sqrt{9{\cdot}8\times6400\times10^3}\,(1{\cdot}414-1)$$

$$=3{\cdot}279\times10^3\text{ m/s.}$$ **Ans.**

17. Let x_0 be the distance from the earth.

Then $\dfrac{Gm\triangle m}{x_0^2}=\dfrac{GM\triangle m}{(d-x_0)^2}$, where $\triangle m$ is the mass of the body.

Hence $\quad x_0=\dfrac{d}{1+\sqrt{\dfrac{M}{m}}}$. **Ans.**

18. E_A = energy of earth + sattelite $A = -\frac{GM_em}{r} + \frac{1}{2}mv^2$.

By dynamics of circular motion

$$\frac{GM_em}{r^2} = \frac{mv^2}{r},$$

$$\therefore \quad E_A = -\frac{1}{2}\frac{GM_em}{r}.$$

Similarly, $$E_B = -\frac{1}{2}\frac{GM_em}{r}$$

$$\therefore \quad E_A + E_B = -\frac{GM_em}{r}. \quad \text{Ans.}$$

By 'conservation of tangential momentum' we have

$$mv + (-mv) = 2mv'$$

or $$v' = 0.$$

$\therefore$ E (energy of the system) = gravitational energy only

$$= -\frac{GM_e \times 2m}{r}$$

$$= -\frac{2GM_em}{r}. \quad \text{Ans.}$$

Since the velocity is zero, the combined body falls directly down. Ans.

19. U (potential energy) $= -\int_r^\infty \frac{kdr}{r^2} = -\frac{k}{r}$.

By conservation of energy we have

$$-\frac{k}{a} + \frac{1}{2}mv^2 = -\frac{k}{x} + \frac{1}{2}mv'^2.$$

By conservation of angular momentum at the extreme position where velocities are at right angles to the position vectors (moment of momentum is also angular momentum)

$$(mv)a = (mv')x \quad \text{or} \quad va = v'x$$

$$\therefore \quad -\frac{k}{a} + \frac{1}{2}mv^2 = -\frac{k}{x} + \frac{1}{2}m\frac{v^2a^2}{x^2}.$$

Now $$v^2 = \frac{k}{2ma} \text{ (given)}.$$

$$\therefore \quad -\frac{k}{a}+\frac{1}{2}m.\frac{k}{2ma}=-\frac{k}{x}+\frac{1}{2}m\frac{k}{2ma}\cdot\frac{a^2}{x^2}$$

$$-\frac{1}{a}+\frac{1}{4a}=-\frac{1}{x}+\frac{a}{4x^2}$$

$$\text{or} \quad \frac{a}{4x^2}-\frac{1}{x}+\frac{3}{4a}=0$$

$$\text{or} \quad \frac{4a^2}{4x^2}-\frac{4a}{x}+3=0$$

$$\text{or} \quad \frac{a^2}{x^2}-\frac{4a}{x}+3=0$$

$$\text{or} \quad \left(\frac{a}{x}-2\right)^2=1$$

$$\text{or} \quad \frac{a}{x}=3 \text{ or } 1$$

$$\text{or} \quad a=x \quad \text{or} \quad a=3x$$

$$\therefore \quad x=\frac{a}{3}. \text{ Ans.}$$

Pendulum

1. Since inside a mine g decreases, T increases and so the pendulum loses. Let it lose n seconds. Then

$$2=2\pi\sqrt{\frac{l}{10}}$$

and $\quad 2\times\frac{86400}{86400-n}=2\pi\sqrt{\frac{l}{g}}$ where g = acceleration inside the mine.

Dividing, $\quad \frac{86400-n}{86400}=\sqrt{\frac{g}{10}}.$

Now, $\quad 10=\frac{GM}{R^2}; \quad g=\frac{GM}{R^3}\ (R-h)$

$$\therefore \quad \frac{g}{10}=\frac{R-h}{R}$$

$$\therefore \quad 1-\frac{n}{86400}=\sqrt{1-\frac{h}{R}}=1-\tfrac{1}{2}\ \frac{h}{R}\ (\because\ h<<R)$$

or $$h=\frac{2nR}{86400}$$

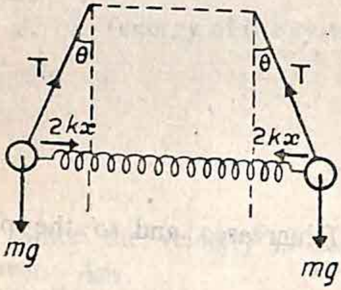
or $$n=\frac{86400\times h}{2R}=\frac{86400\times 200}{2\times 6400\times 100}=1{\cdot}35 \text{ s.} \qquad \textbf{Ans.}$$

2. When a car moves horizontally with acceleration a, the effective acceleration on the bob is $\sqrt{a^2+g^2}$.

$$\therefore \quad T=2\pi\sqrt{\frac{l}{\sqrt{a^2+g^2}}}.$$

When a car negotiates a curved path it has centripetal acceleration given by $a=\frac{v^2}{R}$.

$$\therefore \quad T=2\pi\sqrt{\frac{l}{\sqrt{g^2+\frac{v^4}{R^2}}}}=2\pi\left(\frac{l^2}{g^2+\frac{v^4}{R^2}}\right)^{\frac{1}{4}}.$$



3. (a) When they are deflected through the same angle in the same direction, there is no deformation of the spring. Hence each pendulum will oscillate as a normal simple pendulum.

$$\therefore \quad T=2\pi\sqrt{\frac{l}{g}}. \quad \textbf{Ans.}$$

(b) Let x be the instantaneous displacement of each pendulum from the mean position. Therefore, the spring is elongated by $2x$ and the force exerted by it on each pendulum is $k.2x$.

Unbalanced force on each bob $=2kx+T\sin\theta$ in the negative direction of x

$$=-(2kx+T\sin\theta) \text{ in the +ve direction}$$
$$=-(2kx+T\theta) \text{ when } \theta \text{ small.}$$

As there is no vertical acceleration,

$$T\cos\theta=mg$$

or $$T=mg \quad (\because \cos\theta=1 \text{ where } \theta \text{ small}).$$

$\therefore$ Unbalanced force on each bob $=-(2kx+mg\theta)$

$$=-\left(2kx+\frac{mg}{l}x\right) \quad (\because x=l\theta)$$

By Newton's law of motion

$$-\left(2k+\frac{mg}{l}\right)x=m\frac{d^2x}{dt^2}$$

or
$$\frac{d^2x}{dt^2}=-\left(\frac{2k}{m}+\frac{g}{l}\right)x.$$

Thus $\frac{d^2x}{dt^2}\propto -x$ and so the motion is simple harmonic. Here

$$\left(\frac{2\pi}{T}\right)^2=\frac{2k}{m}+\frac{g}{l}$$

or
$$(2\pi f)^2=\frac{2k}{m}+\frac{g}{l}$$

or
$$f=\frac{1}{2\pi}\sqrt{\frac{g}{l}+\frac{2k}{m}}.\ \textbf{Ans.}$$

4. The velocity of a particle in SHM is given by

$$v=\omega\sqrt{a^2-x^2}$$

$$v_{max}=\frac{2\pi}{\tau}\times a=\frac{2\pi}{2}\times\cdot 05=\cdot 16\text{ m/s. }\textbf{Ans.}$$

At the mean position, from dynamics of circular motion,

$$T-mg=\frac{mv^2_{max}}{l}.$$

$$=\frac{m}{l}\cdot\frac{4\pi^2}{\tau^2}a^2,$$ where τ = Time period.

But $\tau=2\pi\sqrt{\frac{l}{g}}$, or $\tau^2=\frac{4\pi^2 l}{g}$.

Substituting the value of l,

$$\therefore\quad T-mg=\frac{16\pi^4a^2m}{\tau^4 g}$$

or
$$T=mg\left(1+\frac{16\pi^4a^2}{\tau^4g^2}\right)$$

$$=50\times 10^{-3}\times 9\cdot 8\left(1+\frac{16\pi^4\times\cdot 05^2}{2^4\times 9\cdot 8^2}\right)$$

$$=\cdot 49\left(1+\frac{\pi^4\times\cdot 05^2}{9\cdot 8^2}\right)=\cdot 49\text{ newton. }\textbf{Ans.}$$

5. At the surface of the earth, $2=2\pi\sqrt{\frac{l}{g}}$.

At the height of the balloon, $2\times\frac{86400}{86400-400}=2\pi\sqrt{\frac{l}{g'}}$.

Dividing, $\left(1-\frac{1}{216}\right)=\sqrt{\frac{g'}{g}}$.

But, $g=\frac{GM}{R^2}$ and $g'=\frac{GM}{(R+h)^2}$.

$$\therefore\quad \frac{g'}{g}=\left(\frac{R}{R+h}\right)^2$$

$$\therefore\ 1-\frac{1}{216}=\sqrt{\left(\frac{R}{R+h}\right)^2}=\frac{R}{R+h}$$

or $\frac{215}{216}=\frac{R}{R+h}$, or $R=215\,h$

or $h=\frac{R}{215}=\frac{6450\text{ km}}{215}=30$ km. **Ans.**

Elasticity

1. Thermal change in length $=\triangle l=l\alpha\triangle\theta$ where l is the original length, α = expansivity or coefficient of linear expansion and $\triangle\theta$ = change in temperature.

The supports prevent this change by exerting force. So thermal expansion is equal to the elastic extension.

$$\therefore\quad \text{strain}=\frac{l\alpha\triangle\theta}{l}=\alpha\triangle\theta.$$

(Note : Always remember that the strain corresponding to change in temperature change by $\triangle\theta$ is $\alpha\triangle\theta$.)

If A is the area of cross-section and T is the tension, then stress

$$=\frac{T}{A}.$$

$$\therefore\ Y=\frac{\frac{T}{A}}{\alpha\triangle\theta}=\frac{T}{A\alpha\triangle\theta}$$

or $T=YA\alpha\triangle\theta$

$=21\times10^{10}\times\cdot5\times10^{-6}\times12\times10^{-6}\times20$

$=25{\cdot}2$ newton. **Ans.**

2. Obviously there is contraction (thermal) by, say, $\triangle l$ and elastic expansion by, say, $\triangle l'$. Then change in length $= \triangle l - \triangle l'$.

$\triangle l$, Thermal contraction $= l\alpha\triangle\theta = 3\times 12\times 10^{-6}\times 200 = {\cdot}0072$ m.

$$\text{Elastic} \qquad \text{strain} = \frac{\triangle l'}{l} = \frac{\triangle l'}{3}$$

$$\text{stress} = \frac{10g}{\pi({\cdot}5\times 10^{-3})^2} = \frac{10g}{25\pi\times 10^{-8}}.$$

$$\therefore \quad 1{\cdot}995\times 10^{11} = \frac{\dfrac{10g}{25\pi\times 10^{-8}}}{\dfrac{\triangle l'}{3}}$$

or
$$\triangle l' = \frac{30\times 9{\cdot}8}{25\pi\times 1{\cdot}995\times 10^{3}} = {\cdot}0019 \text{ m}$$

$\therefore$ change in length $= {\cdot}0072 - {\cdot}0019 = {\cdot}0053$ m (contraction). Ans.

3. We have,
$$Y = \frac{\dfrac{T}{A}}{\dfrac{\triangle l}{l}} = \frac{T\times l}{A\times\triangle l}$$

or $\qquad T\times l = YA\triangle l.$

Here l, $\triangle l$ and A are the same. $\quad\therefore \quad T\propto Y$

or
$$\frac{T_{steel}}{T_{copper}} = \frac{Y_{steel}}{Y_{copper}}$$

or $\qquad T_{steel} = 2T_{copper} \qquad (\because \quad Y_{steel} = 2\times Y_{copper}$ given).

From 'equilibrium of the plank'
$$T_{steel} + 2T_{copper} = 100$$

or
$$\left.\begin{aligned} T_{copper} &= 25 \text{ kgf} \\ T_{steel} &= 50 \text{ kgf} \end{aligned}\right\} \text{Ans.}$$

4. Let us use the suffix 's' for steel and 'c' for copper. Copper expands more than steel. So increase in temperature will lead to elastic compression of the copper plates and tension of the steel plates. Let F be the force with which the copper plate is contracted.

Thermal expansion of copper plate $= l\alpha_c\triangle\theta$

$$\text{elastic contraction} = \frac{F\times l}{Y_c\times A}.$$

$$\therefore \quad \text{Net expansion of each copper plate} = l\alpha_c\triangle\theta - \frac{Fl}{Y_cA}.$$

Considering 'equilibrium of the rivet' we have $F = 2F'$ where F' is the force on each steel plate.

So each steel plate is expanded by $F/2$,
Thermal expansion of each steel plate $= l\alpha_s \triangle\theta$.

$$\text{Elastic expansion} = \frac{(F/2)\times l}{Y_s \times A}$$

$$\therefore \quad \text{Net expansion} = l\alpha_s \triangle\theta + \frac{Fl}{2Y_sA}.$$

Since length is the same for all the plates we have

$$l\alpha_c \triangle\theta - \frac{Fl}{Y_cA} = l\alpha_s \triangle\theta + \frac{Fl}{2Y_sA}$$

or
$$\alpha_c \triangle\theta - \frac{F}{Y_cA} = \alpha_s \triangle\theta + \frac{F}{2Y_sA}$$

or
$$(\alpha_c - \alpha_s)\triangle\theta = F\left(\frac{1}{2Y_sA} + \frac{1}{Y_cA}\right)$$

or
$$F = \frac{2(\alpha_c - \alpha_s)A\triangle\theta Y_s Y_c}{Y_c + 2Y_s}.$$

Putting
$$Y_s = 2Y_c,$$
$$F = \tfrac{2}{5}(\alpha_c - \alpha_s)A\triangle\theta Y_s$$
$$= \tfrac{2}{5}(18-11)\times 10^{-6}\times 50\times 10^{-4}\times 20\times 2\times 10^{11}$$
$$= 2\times 7\times 4\times 10^3 = 56\times 10^3 \text{ newton. } \textbf{Ans.}$$

5. Let $2l$ be the distance between the supports and let the mid-point go down by x. Suppose $2l'$ is the length of the wire in the displaced position. For equilibrium of the mass we have

$$W = T\sin\theta + T\sin\theta$$

where T = tension and θ is the angle between the initial position of the wire and displaced position

or
$$W = 2T\theta \qquad (\because \ \sin\theta = \theta \text{ when } \theta \text{ small})$$

$$\text{Stress} = \frac{T}{A}, \ \text{strain} = \frac{2(l^2+x^2)^{\frac{1}{2}} - 2l}{2l} = \frac{1}{2}\frac{x^2}{l^2} \ (\because \ x \ll l)$$

$$\because \ \text{Stress} = E\times\text{strain}, \ \frac{T}{A} = E\times\frac{1}{2}\frac{x^2}{l^2} \ \text{or} \ \frac{W}{2A\theta} = \frac{1}{2}\frac{Ex^2}{l^2}$$

or
$$\frac{Wl}{Ax} = \frac{Ex^2}{l^2} \quad (\because \ x = l\theta)$$

or
$$x = \left(\frac{Wl^3}{EA}\right)^{\frac{1}{3}} = l\sqrt[3]{\frac{Mg}{EA}} \qquad (\because \ W = Mg). \ \textbf{Ans.}$$

6. Consider an element of length l subtending an angle $2\triangle\theta$ at the centre. Then $l = (2\triangle\theta)\times R$ where R = radius of the ring. If T is

the tension of the ring then centripetal force

$$=2T\sin\triangle\theta=2T\triangle\theta=T\frac{l}{R}.$$

$$\therefore \quad T\frac{l}{R}=\frac{(\pi r^2 l\rho)v^2}{R} \text{ where } r=\text{radius of cross-section of the ring.}$$

$$\text{or} \quad \sigma\ (\text{stress})=\frac{T}{\pi r^2}=\rho v^2 \quad \text{or } v=\sqrt{\frac{\sigma}{\rho}}=\sqrt{\frac{2\times 10^7}{11300}}=42\text{ms}^{-1}\textbf{Ans.}$$

7. Let F be the force with which they interact. Thermal expansion of rod $1=l_1\alpha_1 t$ and its elastic contraction

$$=\frac{Fl_1}{\alpha_1 E_1}\left(\because \text{ change in length}=\frac{Fl}{\alpha E}\right)$$

$$\therefore \quad \triangle l_1 \text{ (net increase in length of rod 1)}=l_1\alpha_1 t-\frac{Fl_1}{E}$$

$$\text{similarly} \qquad \triangle l_2=l_2\alpha_2 t-\frac{Fl_2}{E_2 A}$$

$$\therefore \quad l_1\alpha_1 t-\frac{Fl_1}{E_1 A}+l_2\alpha_2 t-\frac{Fl_2}{E_2 A}=0$$

$$\text{or} \qquad (l_1\alpha_1+l_2\alpha_2)t=F/A\left(\frac{l_1}{E_1}+\frac{l_2}{E_2}\right)$$

$$F=At.\ \frac{\alpha_1 l_1+\alpha_2 l_2}{\frac{l_1}{E_1}+\frac{l_2}{E_2}}.\ \textbf{Ans.}$$

8. We have, u (energy density) $=\frac{1}{2}$ stress $\times$ strain

$$=\tfrac{1}{2}\times\text{stress}\times\frac{\text{stress}}{K}$$

$$\left(\because K=\frac{\text{stress}}{\text{strain}}\right)$$

$$=\tfrac{1}{2}\beta\ (\text{stress})^2\left(\because\ \beta=\frac{1}{K}\right)$$

$$=\tfrac{1}{2}\beta\ (\rho g h)^2$$

$$\text{or} \qquad u=\tfrac{1}{2}\beta\ (\rho g h)^2$$

$$=\tfrac{1}{2}\times(5\times 10^{-10})\times(1000\times 9{\cdot}8\times 1000)^2$$

$$=2{\cdot}5\times 9{\cdot}8^2\times 10^2=2{\cdot}4\times 10^4\ \text{Jm}^{-3}.\ \textbf{Ans.}$$

9. We have, $u = \frac{1}{2} \times \text{stress} \times \text{strain}$

$$= \tfrac{1}{2} \times E \times \text{strain} \times \text{strain} \quad \left(\because \quad E = \frac{\text{stress}}{\text{strain}}\right)$$

$$= \tfrac{1}{2} E \epsilon^2.$$

U (total energy) = volume × density

$$= \frac{m}{\rho} \times \tfrac{1}{2} \epsilon^2 E = \tfrac{1}{2} \epsilon^2 \frac{m}{\rho} E$$

or $$U = \tfrac{1}{2} \times (10^{-3})^2 \times \frac{5}{7800} \times 2 \times 10^{11}$$

$$= \frac{5}{78} \times 10^3 = 64 \text{ Joule. Ans.}$$

10. See Ex. 8. $p = 2 \frac{\triangle r}{r} \sigma$

$$= 2 \times \frac{\cdot 2}{10} \times 5 \times 10^7 = 2 \times 10^6 \text{ Pa. Ans.}$$

11. See Ex. 8. $p = \frac{\triangle r}{r} . \sigma = \frac{\cdot 05}{5} \times 3 \times 10^8 = 3 \times 10^6$ Pa. Ans.

12. We have, $n = \frac{l}{2l} \sqrt{\frac{T}{m}}$.

Here, $T = \pi r^2 \times \left(Y \times \frac{\triangle l}{l}\right)$ ($\because$ stress $= Y \times$ strain)

and $m = \pi r^2 \rho$

$$\therefore \quad n = \frac{1}{2l} \sqrt{\frac{\pi r^2 Y \frac{\triangle l}{l}}{\pi r^2 \rho}} = \frac{1}{2l} \sqrt{\frac{Y \triangle l}{\rho l}}$$

or $$n = \sqrt{\frac{Y \triangle l}{4 \rho l^3}}$$

$$= \sqrt{\frac{9 \times 10^{10} \times 5 \times 10^{-4}}{4 \times 9000 \quad 13}} = 35 \text{ Hz. Ans.}$$

13. See solution of problem 5. We have there

$x = l \left(\frac{Mg}{EA}\right)^{\frac{1}{3}}$, where l = half the length of the wire

$$=1\left(\frac{\frac{1}{4}\times9\cdot8}{2\times10^{11}\times\pi\times5^2\times10^{-8}}\right)^{\frac{1}{3}}$$

$$=\left(\frac{\cdot25\times9\cdot8}{50\pi\times10^3}\right)^{\frac{1}{3}}=2\cdot5\times10^{-2}\text{ m. } \textbf{Ans.}$$

14. At the lowest position, $T-mg=m\omega^2 l$

$$=m\omega^2(l_0+\triangle l)$$

or $$T=mg+m\omega^2(l_0+\triangle l).$$

We have, stress $=Y\times$ strain $=Y\,\dfrac{\triangle l}{l_0}$

and $$T=A\times\text{stress}=A\times\frac{Y\triangle l}{l_0}$$

$$\frac{AY\triangle l}{l_0}=mg+m\omega^2\ (l_0+\triangle l)$$

or $$\triangle l=\frac{mg+m\omega^2 l_0}{\left(\dfrac{AY}{l_0}-m\omega^2\right)}=\frac{\cdot5\times9\cdot8+\cdot5\times4\times\cdot6^2}{\left(\dfrac{\cdot05\times10^{-4}\times2\times10^{11}}{\cdot6}-\cdot5\times\cdot6^2\right)}$$

$$=\frac{5\cdot62}{1\cdot67\times10^6}=3\cdot36\times10^{-6}\text{ m. } \textbf{Ans.}$$

15. Centripetal force $=T$ (tension) $=m\omega^2(l_0+\triangle l)$.

But $$T=k\triangle l$$

$$\therefore\quad k\triangle l=m\omega^2(l_0+\triangle l)$$

The radius of the circle $=r=l_0+\triangle l$

$$\therefore\quad \triangle l=r-l_0$$

$$\therefore\quad k(r-l_0)=m\omega^2(l_0+r-l_0)$$

or $$r=\frac{kl_0}{k-m\omega^2}=\frac{kl_0}{k-4\pi^2mf^2}.$$ **Proved.**

$$T=m\omega^2(l_0+\triangle l)=m\omega^2 r=m\omega^2.\ \frac{kl_0}{k-4\pi^2mf^2}$$

or $$T=\frac{4\pi^2mkl_0f^2}{k-4\pi^2mf^2}.$$ **Proved.**

16. From 'dynamics of circular motion'

$$T-mg=\frac{mv^2}{r}=\frac{mv^2}{l+\triangle l}.$$

But $$T=k\triangle l$$

$$\therefore \quad k\triangle l-mg=\frac{mv^2}{l+\triangle l}$$

or $$k\triangle l(l+\triangle l)-mg(l+\triangle l)=mv^2 \qquad .. \quad \text{(i)}.$$

By 'energy conservation' (point of suspension is taken as reference)

$$0=\tfrac{1}{2}\,mv^2+\tfrac{1}{2}\,k\triangle l^2-mg(l+\triangle l)$$

or $$mv^2+k\triangle l^2=2mg(l+\triangle l) \qquad .. \quad \text{(ii)}.$$

Combining (i) and (ii) to eliminate v

$$k\triangle l(l+\triangle l)-mg(l+\triangle l)=2mg(l+\triangle l)-k\triangle l^2$$

or $k\triangle l\times l=3mgl-k\triangle l^2$ neglecting $\triangle l$ in comparision to l)

or $k\triangle l(l+\triangle l)=3mgl$

or $k\triangle l\times l=3mgl$ (again neglecting $\triangle l$ in comparison to l)

or $\triangle l=\dfrac{3mg}{k}$. **Proved.**

Putting the value of $\triangle l$ in (ii)

$$mv^2=2mg\left(l+\frac{3mg}{k}\right)-k.\,\frac{9m^2g^2}{k^2}$$

$$v^2=2gl+\frac{6mg^2}{k}-\frac{9mg^2}{k}=2gl-\frac{3mg^2}{k}$$

or $$v=\sqrt{2g\left(l-\frac{3mg}{2k}\right)}.$$ **Proved.**

Solutions to Test Questions

1. (i) It will lose. (ii) Yes. Let l' be the length of the thread and 'a' be the distance of cg from the point of suspension. Then

$$T=2\pi\sqrt{\frac{l'+a}{g}}$$

or $$l'+a=\frac{g^2}{4\pi^2}\,T^2,\text{ or } l'=\frac{g^2}{4\pi^2}\,T^2-a.$$

The plot of l' against T^2 is a straight line, the intercept of which on the l'-axis gives a. No, in a space laboratory a simple pendulum will not work.

(iii) The pendulum in which the disc is rigidly fixed to the rod will have greater time period than the one in which the disc is free to turn because the disc moves translationwise, and the system behaves like a simple pendulum.

(iv) In drinking because there is no force to pour water out of the vessel.

(v) Water actually flows from higher potential to the lower potential. The mouth is actually at lower potential than the source.

(vi) No, the component of the stress, normal to the face is the tensile stress and the parallel component is the shearing stress.

(vii) Yes, a liquid cannot withstand shearing stress. As it cannot withstand shearing stress, it has no definite shape. So it is true that a liquid surface is horizontal because it has no definite shape.

(viii) Because the greater the mass, the greater the gravitational pull. $F=\frac{GmM}{R^2}=mg$, or $g=\frac{GM}{R^2}=a$ constant.

No, for pull of an engine, $F=mf$, or $f=\frac{F}{m}$.

The acceleration is inversely proportional to the mass.

(ix) When the engine of the spaceship be shut off and there be no resistance of the external medium.

(x) No, when it is clamped at one end and free to move out at the other end it is not strained. Yes, when clamped at both ends it is prevented from expanding and so it is strained.

2. (i). (b). Shearing stress $=\frac{\text{shearing force}}{\text{area}}=\frac{F\sin\theta}{ab}$.

(ii). (b). shearing stress $=\frac{F\cos\theta}{bc}$.

(iii). (d).

(iv). (b). By Kepler's laws.

(v). (c), : K (bulk modulus) $=\frac{1}{\beta(\text{compressibility})}=\frac{1}{\cdot 5\ \text{GPa}^{-1}}$

or $K=2\ \text{GPa}=2\times 10^9\ \text{Pa}$ $(\because\ 10^9=1\ \text{giga})$.

(vi). (d). : strain $=\alpha\triangle\theta=12\times10^{-6}\times30=36\times10^{-5}$.

(vii). (c). : $W=\frac{1}{2}\times\text{stress}\times\text{strain}=\frac{1}{2}\times Y\times\text{strain}^2$

$$=\tfrac{1}{2}Y\left(\frac{\triangle l}{l}\right)^2$$

$$\therefore\ W\propto\triangle l^2.$$

(viii). (a). : The velocity changed from escape velocity to orbital velocity. $\triangle v=v_e-v_0=\sqrt{\frac{2GM}{R}}-\sqrt{\frac{GM}{R}}$

$$=\sqrt{\frac{GM}{R}}\,(\sqrt{2}-1).$$

(ix). (a).

(x). (a). : U_i (initial potential energy) $= -\dfrac{GM}{R}$ (no kinetic energy)

$$U_f=-\tfrac{1}{2}\,\frac{GM}{2R}=-\frac{GM}{4R},$$

$$\therefore\quad \triangle U=U_f-U_i=\frac{3}{4}\,\frac{GM}{R}=\frac{3}{4}\,gR.$$

3. (a) $mg'=m\omega^2R_0$ or $g'=\omega^2R_0=\left(\dfrac{2\pi}{T}\right)^2R_0.$

Here $T=\dfrac{86400}{15}$ s. $R_0=7\times10^6$ m

$$\therefore\quad g'=\left(\frac{30\pi}{86400}\right)^2\times7\times10^6=7{\cdot}7\ \text{m/s}^2.$$

(b) Binding energy = energy required to disintegrate the system
= present energy of the system

$$=-\tfrac{1}{2}.\,\frac{GMm}{R}$$

$$=-\tfrac{1}{2}\times\frac{6{\cdot}67\times10^{11}\times5\times10^{30}\times250}{6{\cdot}67\times10^{6}}$$

$$=-6{\cdot}25\times10^{15}\ \text{Joule. Ans.}$$

4. When the increase in length produced by tensile force is equal to the thermal contraction, the rod is freed from strain.

elastic strain $=\dfrac{1}{Y}\times\text{stress}=\dfrac{1}{Y}\times\dfrac{F}{A}.$

Thermal strain $=\alpha\triangle\theta$

$$\therefore\quad \frac{F}{YA}=\alpha\triangle\theta,\ \text{or}\ \triangle\theta=\frac{F}{YA\alpha}.$$

$$\therefore\quad \triangle\theta=\frac{10\times9{\cdot}8}{2\times10^{11}\times5\times10^{-6}\times10^{-5}}=9{\cdot}8$$

$\therefore$ Temperature $=27-9{\cdot}8=17{\cdot}2$°C. Ans.

(b) Let f be the force per unit area with which the ring gripes over the wheel. Let a be its width and c be its thickness.

Consider equilibrium of one half of the ring. Inward force $=T+T$ $=2T$ where T is tension produced due to thermal contraction.

Outward force $=(\pi R\times a)\times f$

For equilibrium, $2T=\pi Raf$

Now $T=$ tension $=$ area $\times$ stress $=(a\times c)\times(Y\times \text{strain})$

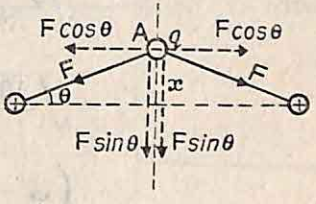
$$=(a\times c)\times Y\times \alpha\triangle\theta$$

$$\therefore\quad (a\times c)\ Y\alpha\triangle\theta=\pi Raf$$

$$f=\frac{Y\alpha\triangle\theta c}{\pi R}$$

$$=\frac{200\times 10^{9}\times 10^{-5}\times 170\times \cdot 01}{\pi\times \cdot 75}=1{\cdot}44\times 10^{6}\ \text{N/m}^2.\ \textbf{Ans.}$$

5. Let x be the instantaneous displacement at right angles to the line joining the two masses. Referring to the figure, unbalanced force on m



$=2F\sin\theta$ in the $-$ve direction of x

$=-2F\sin\theta$ in the $+$ve direction of x

$$=-2\left(\frac{GMm}{a^2+x^2}\right).\ \frac{x}{\sqrt{a^2+x^2}}$$

$$=-\ \frac{2GMmx}{(a^2+x^2)^{\frac{3}{2}}}=-\frac{GMmx}{a^3}\qquad (\because\ x<<a)$$

$$=m\frac{d^2x}{dt^2}$$

$$\therefore\quad \frac{d^2x}{dt^2}=-\frac{GM}{a^3}\ x$$

Thus $\frac{d^2x}{dt^2}\propto -x$ and hence the motion is simple harmonic.

Here $\left(\frac{2\pi}{T}\right)^2=\frac{GM}{a^3}$

or $T=2\pi\sqrt{\frac{a^3}{GM}}$

$$f=\frac{1}{2\pi}\sqrt{\frac{GM}{a^3}}. \text{ Ans.}$$

6. Let ω be the actual angular velocity of the satellite from east to west and ω_e is the angular speed of the earth (west to east)

$$\therefore \quad \omega_{relative}=\omega-(-\omega_e)=\omega+\omega_e$$

or

$$\omega=\omega_r-\omega_e$$

By dynamics of circular motion

$$\frac{GMm}{R^2}=m\omega^2R, \text{ or } \omega^2=\frac{gR_e^2}{R^3} \quad (\because \quad GM=gR_e^2)$$

or

$$\omega=\sqrt{\frac{gR_e^2}{R^3}}$$

or

$$\omega_r=\sqrt{\frac{gR_e^2}{R^3}}+\omega_e$$

$$=\sqrt{\frac{10\times 6{\cdot}4^2\times 10^{12}}{2^3\times 10^{21}}}+7{\cdot}27\times 10^{-5}$$

$$\left(\because \quad \omega_e=\frac{2\pi}{86400}=7{\cdot}27\times 10^{-5}\right)$$

$$\omega_r=22{\cdot}6\times 10^{-5}+7{\cdot}27\times 10^{-5}=30\times 10^{-5} \text{ rad/s.}$$

$$\therefore \quad \frac{2\pi}{\tau}=30\times 10^{-5}$$

or $\tau=2{\cdot}09\times 10^4 \text{ s}=348 \text{ min}=5 \text{ hr } 48 \text{ min}$. **Ans.**

7. (a) See Ex. **5.** $T=2\pi\sqrt{\dfrac{m_1l_1^2+m_2l_2^2+m_3l_3^2}{(m_1l_1+m_2l_2+m_3l_3)g}}$. **Ans.**

(b) $T=2\pi\sqrt{\dfrac{l}{g+a}}=2\pi\sqrt{\dfrac{l}{10+2}}$.

When the elevator is at rest

$$T=2 \text{ s.} \qquad \therefore \quad 2=2\pi\sqrt{\frac{l}{10}}$$

$$\therefore \quad \frac{T}{2}=\sqrt{\frac{10}{12}}, \qquad \text{or} \quad T=1{\cdot}82 \text{ s.} \text{ **Ans.**}$$

8. See Exercise 19.

LESSON 7

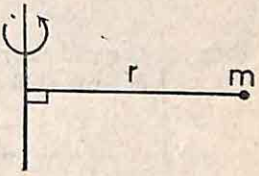
Topics : **Rotational Dynamics**
Moment of Inertia
Conservation of Angular Momentum

7.1 Rotational Dynamics

Torque (τ) . The moment of a force about a rotation axis is called torque. As a vector, it is represented along the normal to the plane of rotation, that is, *along the axis of rotation.*

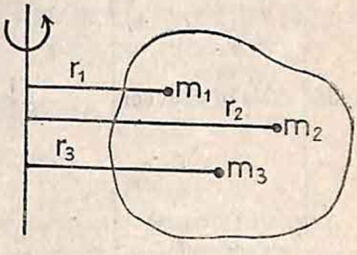
Moment of inertia. The product of the mass of a particle and the square of its distance from the axis of rotation is called its moment of inertia. Thus,

$I = mr^2$ for a particle.



The moment of inertia of a rigid body is the sum of the moments of inertia of its constituent particles.

$$I = m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \dots$$
$$= \Sigma mr^2.$$

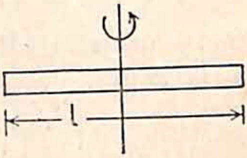
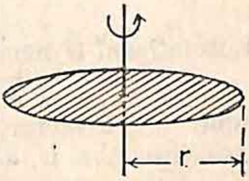
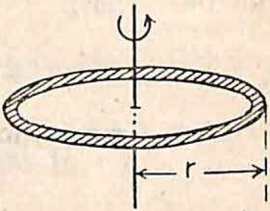
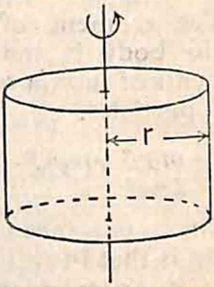
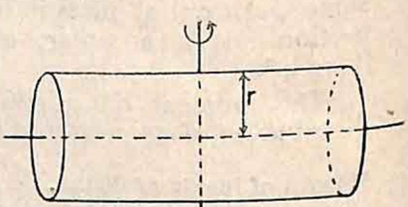


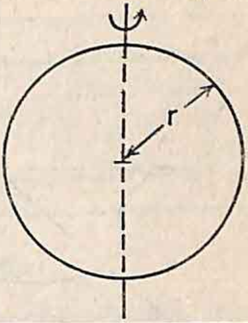
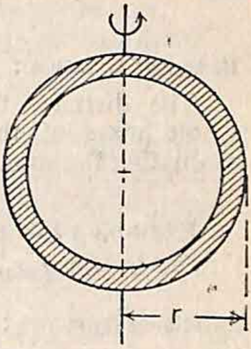
The importance of moment of inertia is that in rotational dynamics, it occupies precisely the same position of mass in linear motion. It is a scalar, and has the dimension ML^2. Its unit is kg m^2.

The moment of inertia of a rigid body depends both on the distribution of its mass and on the position of the axis of rotation.

Moment of Inertia of Bodies.

The moments of inertia of bodies having simple geometrical shapes are given overleaf for axes passing through the centre of mass in each case [m = mass of body].

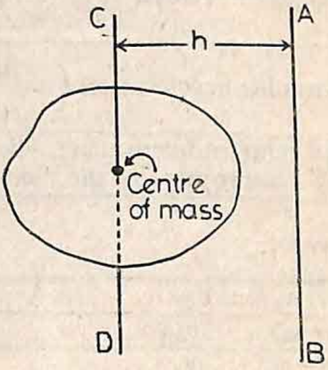
Shape of Body	*Axis of Rotation*	*Moment of Inertia*	*Figure*
1. Thin rod	Perpendicular to length	$ml^2/12$	
2. Disc	Perpendicular to plane of disc	$\frac{1}{2} mr^2$	
3. Ring	Perpendicular to plane of ring	mr^2	
4. Cylinder	Axis of cylinder	$\frac{1}{2} mr^2$	
5. Cylinder	Perpendicular to axis of cylinder	$m[(l^2/12)+(r^2/4)]$	

Shape of Body	*Axis of Rotation*	*Moment of Inertia*	*Figure*
6. Sphere	Diameter	$\frac{2}{5} mr^2$	
7. Spherical shell (thickness of wall is negligible)	Diameter	$\frac{2}{3} mr$	

When the moment of inertia of a body about a particular axis is known, it is sometimes possible to obtain the moment of inertia about some other axis, using the two following theorems :

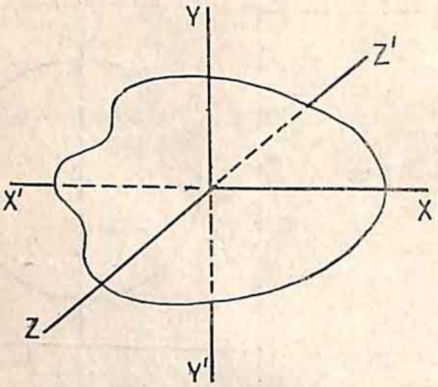
Theorem of Parallel Axis of Moment of Inertia :

$$I = I_{cm} + mh^2$$



where I is the moment of inertia about any axis *(AB)*, I_{cm} is the moment of inertia about a parallel axis *(CD)* through the centre of mass of the body, and *h* is the distance between the axes.

Theorem of Perpendicular Axes :



$$I_z = I_x + I_y$$

where z-axis is normal to the surface of the body and x and y axes are in the surface of the body. This theorem holds only for laminar bodies.

Radius of Gyration :

The distance from the axis of rotation of the point where the whole mass of the body may be supposed to be concentrated for evaluating the moment of inertia of the body. Thus,

$I = mk^2$ where k = radius of gyration.

Example : For a disc rotating about its axis,

$$I = mr^2/2 = mk^2 \quad \text{or} \quad k = r/\sqrt{2}.$$

Rotational Dynamics :

Rotational dynamics is exactly the same as the dynamics of linear motion, the corresponding quantities being as follows :

Rotational Motion	*Linear Motion*
Moment of inertia (I)	Mass (m)
Torque (τ)	Force (F)
Angular displacement (θ)	Displacement (s)
Angular velocity $\left(\omega = \frac{d\theta}{dt}\right)$	Velocity (v)
Angular acceleration $\left(\alpha = \frac{d\omega}{dt}\right)$	Acceleration (a)
Angular momentum ($L = I\omega$) = moment of momentum	Momentum ($p = mv$)

Kinematics :

$\omega = \omega_0 + \alpha t$	$v = u + at$
$\theta = \omega_0 t + 1/2\alpha t^2$	$s = ut + \frac{1}{2}at^2$
$\omega^2 - \omega_0^2 = 2\alpha\theta$	$v^2 - u^2 = 2as$

Equation of Motion :

$\tau = I\alpha$	$F = ma$

Kinetic Energy :

$T = \frac{1}{2} I\omega^2$	$T = \frac{1}{2} mv^2$

Work Done :

$W = \int \tau . d\theta$	$W = \int F . ds.$

7.2 Principle of Conservation of Angular Momentum

When no external torques act on a system of rotating bodies, the algebraic sum of their angular moments remains constant.

That is, $L =$ a constant when free from externol torques.

7.3 Rolling without Slipping

When a body rolls in such way that the distance through which its centre of mass moves, any point on its circumference also moves through the same distance. If x is the distance through which its centre of mass moves and θ is the angle through which a point on the circumference moves then we say the body is rolling without slipping if

$$x = \theta . r \quad \text{or} \quad v_{cm} = \omega r.$$

This also means that the point of contact *has no velocity at any* instant and it becomes *instantaneous centre of rotation.*

CRITICAL QUESTIONS

1. Can the mass of a body be considered as concentrated at its centre of mass for purpose of calculating its rotational inertia ?

Ans. No.

2. If two circular discs of the same weight and thickness are made from metals having different densities, which disc will have the larger moment of inertia about its central axis ?

Ans. We have $I_1 = \frac{1}{2} mr_1^2$ and $I_2 = \frac{1}{2} mr_2^2$.

$$\frac{I_1}{I_2} = \frac{r_1^2}{r_2^2}. \quad \text{Now } m = \pi r_1^2 t \rho_1 = \pi r_2^2 t \rho_2; \quad \therefore \quad \frac{r_1^2}{r_2^2} = \frac{\rho_2}{\rho_1}$$

$$\therefore \quad \frac{I_1}{I_2} = \frac{\rho_2}{\rho_1}. \quad \text{Thus} \quad I \propto \frac{1}{\rho}.$$

Thus the one having greater density will have less rotational inertia.

3. About what axis would a uniform cube have its minimum rotational inertia ?

Ans. About a diagonal, because about a diagonal the mass is more concentrated.

4. Consider four bodies : a ring, a cube, a disc and a sphere. All the bodies have the same mass. The ring, disc and sphere have the same diameter, equal to the length of the cube on each edge. All rotate about their axes through their respective centres of mass. Which one has the largest moment of inertia ? Which the smallest ?

Ans. I (ring) $=\frac{1}{4}ma^2$ where a is the length of each edge of the cube, I (cube) $=\frac{1}{6}ma^2$, I (disc) $=\frac{1}{8}ma^2$, I (sphere) $=\frac{1}{10}ma^2$.

$$I_1 : I_2 : I_3 : I_4 = 1/4 : 1/6 : 1/8 : 1/10 = 30 : 20 : 15 : 12.$$

Thus the ring has got the greatest moment of inertia and the sphere the smallest.

5. A person can distinguish between a raw egg and a hard boiled one by spinning each one on the table. Explain.

Ans. The hard boiled egg will continue to spin for longer period because of its greater moment of inertia.

6. A wooden sphere rolls down two different inclined planes of the same height but different inclines. Will it reach the bottom with the same speed in each case ? Will it take longer to roll down one incline than the other ? If so, which one and why ?

Ans. The acceleration down an inclined plane is given by

$$a=\frac{g \sin \alpha}{1+\frac{k^2}{r^2}}$$ where k is the radius of gyration.

h (height of the plane) $= s \sin\alpha$ where s is the length of the plane.

Now, $$s=\frac{1}{2}at^2=\frac{1}{2}\frac{g \sin\alpha}{1+\frac{k^2}{r^2}}t^2$$

or $$\frac{h}{\sin\alpha}=\frac{1}{2}\frac{g \sin\alpha}{1+\frac{k^2}{r^2}}t^2, \quad \text{or} \quad t^2 \sin^2\alpha = \text{a constant}$$

or $$t \propto \frac{1}{\sin\alpha}.$$

Thus the greater the time of descent the less is the inclination to the horizontal.

$$v^2=2as=2\cdot\frac{g \sin\alpha}{1+\frac{k^2}{r^2}}\cdot\frac{h}{\sin \alpha}$$

$$v^2 = \text{a constant}, \qquad \text{or} \qquad v = \text{a constant}.$$

Thus it will reach the bottom with the same speed whatever be the inclination.

7. A man, with two equal masses held at arms length, stands on a rotating table. If he moves the masses and puts them over his shoulders, does this change his speed of rotation ? Explain.

Ans. The only external forces acting on the system (man+table) is gravity and reaction of the ground, and those exert no torque about the axis of rotation. Hence the angular momentum of the system is conserved about this axis. When the man puts the masses on his shoulder, the moment of inertia of the system about the axis decreases and so the angular speed of the system increases.

8. A student stands on a platform that can rotate only about a vertical axis. In his hand he holds the axle of a rim-loaded bicycle wheel with its axis vertical. The wheel is spinning about the vertical axis with an angular speed ω_0, but the student and the platform are at rest. The student turns the wheel through 180°, that is, the axle of the wheel is held downward by the student. What happens ?

Ans. The platform starts rotating with double the initial angular momentum of the wheel. To justify it let us consider 'wheel+student+platform' as our system. The initial angular momentum of the system is $I_0\omega_0$, where I_0 is the moment of inertia of the wheel about the vertical axis. Since there is no external torque on the system about the vertical axis, the angular momentum about the vertical axis must be conserved about this axis. Let I be the moment of inertia of 'student+platform' about the vertical axis. The angular momentum of the wheel about the vertical axis is now $-I_0\omega_0$.

Let ω be the angular velocity of the platform. Then by the principle of angular momentum conservation

$$I_0\omega_0=-I_0\omega_0+I\omega, \qquad \text{or} \qquad I\omega=2I_0\omega_0$$

9. A diver can make several somersaults before striking water. Explain.

Ans. Let us consider diver as a system and his limbs as its components. When he leaves the diving board he has a certain angular speed ω_0 about the horizontal axis through his centre of mass. Now there are no external forces acting on him except gravity, and gravity exerts no torque about his centre of mass. His angular momentum remains constant, and $I_0\omega_0=I\omega$. When he pulls in his limbs, his moment of inertia decreases and consequently his angular velocity increases.

10. Show that in course of diving the rotational kinetic energy of a diver, taking somersaults before striking water, increases. What is the source of this increased energy ?

Ans. Since the diver is free from external torque about horizontal axis through his centre of mass, his angular momentum remains constant. Therefore $I_0\omega_0=I\omega$ or $I_0^2\omega_0^2=I^2\omega^2$ .. (i).

When he pulls in his limbs, his moment of inertia about horizontal axis decreases.

$$\therefore \quad I<I_0 \quad \text{or} \quad \frac{1}{I}>\frac{1}{I_0}; \quad \text{or} \quad \frac{1}{I}\times I^2\omega^2>\frac{1}{I_0}\cdot I_0{}^2\omega_0{}^2$$

$$\text{or} \quad \tfrac{1}{2}I\omega^2>\tfrac{1}{2}I_0\omega_0{}^2.$$

The source of this increased kinetic energy is the diver himself, who does work when he pulls the parts of his body together. His body energy is converted into his kinetic energy.

11. If the polar ice caps melts and spreads uniformly, how will the length of the day be affected ?

Ans. When ice of poles melts the water formed would spread on the surface of the earth and so its moment of inertia would be increased. By conservation law of angular momentum, the speed of rotation would be reduced, that is, the length of the day would be increased.

12. A circular turntable rotates at constant angular velocity about a vertical axis. There is no friction and no driving torque. An ice-pan containing ice also rotates along with it. The ice melts but none of the water escapes from the pan. Is the velocity now greater, the same as, or less than the original velocity ? Give reasons for your answer.

Ans. Due to accumulation of mass near the edge, moment of inertia of the system increases and so angular velocity of the system decreases.

13. In order to get a billiard ball to roll without sliding from the start the cue must hit the ball not at the centre but exactly at height $\frac{2}{5}R$ above the centre where R is the radius of the sphere. Explain this.

Ans. Let the ball be struck by a horizontal force F at a height d above the centre. Then torque exerted by it on the ball about its centre is $F.d$. By the formula $\tau=I\alpha$, we have $F.d=\frac{2}{5}MR^2\times\alpha$.

Let a be the acceleration of the centre of mass produced by this force. Then $F=Ma$. For rolling without sliding we have $v=\omega R$ and $a=\alpha.R$; $\therefore$ $F=M\alpha R$.

$$\therefore \quad M\alpha R\times d = \tfrac{2}{5}MR^2\times\alpha, \quad \text{or} \quad d=\tfrac{2}{5}MR.$$

14. A student stands on a turntable with a rim loaded bicycle wheel. He holds the shaft of the wheel vertical and in the beginning there is no motion of the turntable. Now the student is asked to turn the wheel in the clockwise direction relative to him. What happens to the motion of turntable + student ?

Ans. The system (turntable + student) will turn in the opposite direction. Since the entire system is free from external torques, the angular momentum of the system (wheel + turntable + student) remains constant at zero. If I = moment of inertia of turntable + student and I_0 = moment of inertia about the vertical axis of rotation of the table and ω_0 = angular velocity of rotation (relative to the earth) then

$$I_0\omega_0 + I\omega = 0$$

or

$$\omega = -\frac{I_0\omega_0}{I}.$$

Note. This is a case similar to recoil of gun when a shot is fired from it. This is called rotational recoil.

15. The melting of polar icecaps is supposed to be a possible cause of the variation of the earth's time of rotation. Explain.

Ans. A liquid of given mass has less inertia for rotational motion than the same mass of solid because a liquid cannot sustain shearing force. It always tends to move translationwise. Hence melting of icecaps results in reduction of rotational inertia of the earth and hence angular speed of rotation of the earth increases or time of rotation decreases. If water spreads uniformly, the result will be as in Ex. 11.

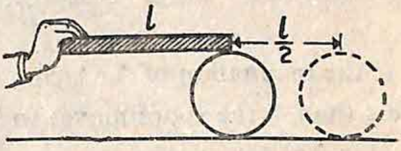
16. Many rivers flow toward the equator. What effect does the sediment they carry to the seas have on the rotation of the earth ?

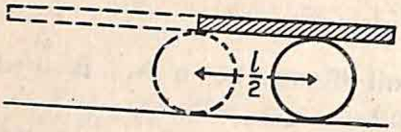
Ans. The sediments are carried away from the axis of rotation and so there is increase of rotational inertia of the earth by this action of the rivers. So the rotational speed is decreased.

17. A student holds a heavy, rim-loaded rapidly spinning bicycle wheel in the horizontal direction. If he is asked to tilt the shaft upward a little, in what direction does he experience torque due to the spinning wheel ?

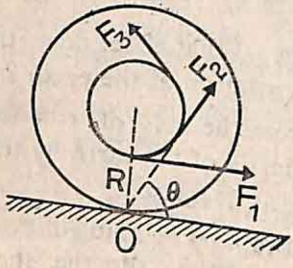
Ans. The angular momentum as a vector is always perpendicular to the plane of rotation, so in the beginning the angular momentum vector is horizontal. When he tilts a little, the angular momentum of the same magnitude makes a small angle with the horizontal. The change in L is vertical upward. The corresponding torque would tend to turn the shaft to the left of the student. So he must exert a torque to his right to keep the shaft in the tilted position.

18. A cylindrical drum, pushed along by a board from an initial position shown in the figure, rolls forward on the ground a distance $l/2$, equal to half the length of the board. There is no slipping at any contact. Where is the board then ? How far has the man walked ?





Ans. Since there is no slipping, the points of contact are instantaneous centres of rotation of the cylinder. Thus, if x is the displacement of the *centre of mass* (and therefore of the ends as well) of the board, the displacement of the centre of the cylinder is $x/2$. So if the centre of the cylinder is displaced by $\frac{1}{2}l$, then $x/2 = \frac{1}{2}l$ or $x = l$. Thus the board will be displaced by l. The position of the board is as shown in the figure. The man walks through a distance l.



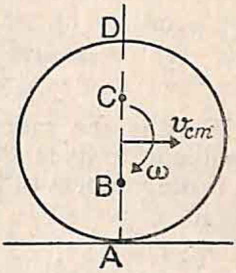
19. A yo-yo (a double disc system joined by a short light tube at the centre) is resting on a horizontal table over which it can roll without slipping. A string wound several round on the short tube is pulled by a force. Three directions of the force are shown in the figure. F_1 is horizontal to the right, F_2 tangential to the tube and passing through the point of contact O and F_3 at inclination greater than that of F_2 with the horizontal. What is the direction of motion in each case ?

Ans. When a body rolls without slipping, the point of contact is the instantaneous centre of rotation. When the force passes through the point of contact, the moment of the forces about the instantaneous centre 'O' is zero. Therefore, if the spool was at rest initially, it will not rotate about O and, consequently, it will not move translationally.

Let θ be the inclination of the thread in the second position with the horizontal. Then $\cos\theta = \frac{r}{R}$.

If the inclination of the string with the horizontal (such as F_1) is less than θ, the spool moves to the right and if greater than θ, the spool moves to the left.

20. A sphere of radius R rolls without slipping on a horizontal road A, B, C and D are four points on the vertical line through point of contact. What are their translational velocities ? Velocity of the centre of mass is v_{cm}.



Ans. The velocity of any point of body 'rolling + sliding' is the vector sum of the velocity of the centre of mass plus the velocity due to rotational motion.

$\therefore \quad v_A = v_{cm} - \omega R = 0 \qquad (\because \; v_{cm} = \omega R$ when there is no slip)

$$v_B = v_{cm} - \frac{\omega R}{2} = v_{cm} - \frac{v_{cm}}{2} = \frac{1}{2} v_{cm}.$$

$$v_C = v_{cm} + \frac{\omega R}{2} = v_{cm} + \frac{v_{cm}}{2} = \tfrac{3}{2} v_{cm}.$$

$$v_D = v_{cm} + \omega R = v_{cm} + v_{cm} = 2v_{cm}.$$

21. If a string is wound round a cylinder and it leaves the cylinder tangentially at A, what is the speed at which the string leaves the cylinder ?

Ans. It leaves the cylinder with the speed of the point A which is $2v_{cm}$ when there is no slip.

OBJECTIVE QUESTIONS

1. A rod of length 1 m and mass $\frac{1}{2}$ kg rotates at angular speed 6 rad/sec about one of its ends. The kinetic energy of the rod is

(a) 1 joule, (b) 2 joule, (c) 3 joule, (d) 4 joule.

Ans. (c). $T = \frac{1}{2} I\omega^2 = \frac{1}{2}(\frac{1}{3} ml)^2 \omega^2 = 1/6 \times \frac{1}{2} \times 1^2 \times 6^2 = 3$ joule.

[Use theorem of parallel axes to obtain $I = \frac{1}{3} ml^2$ about one end.]

2. A sphere of mass 2 kg and radius 5 cm rolls without slipping along a horizontal plane. The velocity of its centre of mass is 10 cm/sec. The kinetic energy of the sphere is

(a) 0·07 J, (b) 0·014 J, (c) 0·025 J, (d) 1·25 J.

Ans. (b). $T = \frac{1}{2} I\omega^2 + \frac{1}{2} mv^2$

$I = \frac{2}{5} mr^2$ and $v = \omega r$ as it is rolling without slipping.

$\therefore \quad I = \frac{7}{10} mv^2 = \frac{7}{10} \times 2 \times {\cdot}1^2 = {\cdot}014$ J.

3. The torque required to stop a wheel 5×10^{-3} kgm² from a speed of 20 rad/s in 10 s is

(a) 1 Nm, (b) ·1 Nm, (c) ·01 Nm, (d) ·001 Nm.

Ans. (c) From $\omega = \omega_0 + \alpha t, \quad 20 = 0 + \alpha.10 \quad$ or $\alpha = 2$ rad/s².

From $\tau = I\alpha, \quad \tau = 5 \times 10^{-3} \times 2 = {\cdot}01$ Nm.

4. A sphere of mass 10 kg and radius ·5 m rotates about a tangent. The moment of inertia of the sphere is

(a) 5 kgm², (b) 2·7 kgm², (c) 3·5 kgm², (d) 4·5 kgm².

Ans. (c). $I_{tangent}=\frac{2}{5}Mr^2+Mr^2=\frac{7}{5}Mr^2=\frac{7}{5}\times10\times\cdot5\times\cdot5$

$$=3\cdot5 \text{ kgm}^2.$$

5. If I_1 is the moment of inertia of a thin rod about an axis perpendicular to its length and passing through its centre of mass and I_2 is the moment of inertia of the ring formed by bending the rod, then

(a) $I_1 : I_2=1:1$, (b) $I_1 : I_2=\pi^2:3$,
(c) $I_1 : I_2=\pi:4$, (d) $I_1 : I_2=3:5$.

Ans. (c). $I_1=\frac{1}{12}ml^2$ and $I_2=m\left(\frac{l}{2\pi}\right)^2$

$$\therefore \quad \frac{I_1}{I_2}=\frac{ml^2}{12}\times\frac{4\pi^2}{ml^2}=\frac{\pi^2}{3}.$$

6. The radius of gyration of a thin rod of length l about an axis passing through one of its ends and perpendicular to it is

(a) $l/\sqrt{2}$, (b) $\frac{l^2}{3}$, (c) $\frac{l}{\sqrt{3}}$, (d) $\frac{l}{4}$.

Ans. (b).

7. Consider a body consisting of two spherical masses of 8 kg each connected by a light rigid rod 1 m long. Treat the spheres as point particles and neglect the mass of the rod. The moment of inertia of the dumbbell about an axis normal to it through its centre is (a) 16 kgm², (b) 4 kgm², (c) 8 kgm², (d) 12 kgm.

Ans. (c).

EXAMPLES

1. Two masses m_1 and m_2 are connected by an in extensible light thread passing over pulley of mass M and radius R. Calculate the common acceleration of the masses and the tensions in the vertical portions of the string.

Sol. Considering downward motion of m_1 we have

$$T_1-m_1\,g=m_1\,a \quad \dots \text{(i).}$$

Considering upward motion of m_2

$$m_2\,g-T_2=m_2\,a \quad \dots \text{(ii).}$$

Considering 'rotational motion' of the pulley (a disc)

$$\tau=T_2R-T_1R=(\tfrac{1}{2}MR^2)\times\alpha \quad \dots \text{(iii)}$$

where α is angular acceleration of the pulley.

But $$a=\alpha\,.\,R$$

$$\therefore\quad (T_2-T_1)R=\frac{1}{2}MR^2\times\frac{a}{R}$$

or $$T_2-T_1=\tfrac{1}{2}Ma \qquad \text{.. (iv).}$$

Solving (i), (ii) and (iv) for a

$$a=\frac{(m_2-m_1)\,g}{\left(m_1+m_2+\dfrac{M}{2}\right)}.$$

$$T_1=\frac{2m_1m_2+\frac{1}{2}m_1M}{m_1+m_2+\frac{1}{2}M}\text{ and }T_2=\frac{2m_1m_2+\frac{1}{2}m_2M}{m_1+m_2+\frac{1}{2}M}\textit{ Ans.}$$

Note. Note carefully here how on neglecting the mass of the pulley, these formulae are reduced to the simple case which we considered earlier.

*2. A uniform disc of radius R is first spinned about its axis to the angular velocity ω and then carefully placed with its flat face on a horizontal surface. How long will the disc be rotating on the surface if the friction coefficient is equal to μ ?

Sol. Consider a ring of the disc of radius x and thickness dx. The frictional force on the ring

$$=\mu\times(mass)\times g=\mu(2\pi x dx\rho)\,g$$

where ρ is the surface density of mass

$$=\mu\,.\,2\pi x dx.\,\frac{M}{\pi R^2}\,g=\frac{2Mg\mu}{R^2}\,xdx.$$

Torque of this force $=\left(\dfrac{2Mg\mu}{R^2}\,xdx\right)\times x=\dfrac{2Mg\mu}{R^2}\,x^2dx.$

Total torque $=\displaystyle\int_0^R\frac{2Mg\mu}{R^2}\,x^2dx=\frac{2}{3}\,Mg\mu R.$

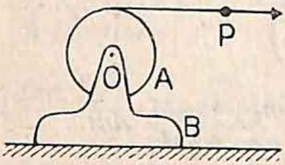
Let α be the angular retardation produced.

Then $\dfrac{2}{3}\,Mg\mu R=\left(\dfrac{1}{2}MR^2\right)\alpha,\quad$ or $\quad\alpha=\dfrac{4}{3}\,\dfrac{\mu g}{R}.$

From $\omega=\omega_0+\alpha t$ we have, $\omega=\frac{4}{3}\frac{\mu g}{R}t$

or
$$t=\frac{3\omega R}{4\mu g}.\quad Ans.$$

*3. A uniform solid cylinder A of mass m_1 can freely rotate about a horizontal axis fixed to a mount B of mass m_2 (see figure). A constant horizontal force is applied to end P of a light thread lightly wound on the cylinder. The friction between the mount and the supporting horizontal surface is supposed to be absent. Find the acceleration of this point P and the kinetic energy of this system t seconds after beginning of motion.



Sol. Let x and x_{cm} be the displacement of the point P and the centre of mass O of the cylinder and θ is the angle of rotation of the cylinder. Then

$$x-x_{cm}=R\theta \quad \text{or} \quad \ddot{x}-\ddot{x}_{cm}=R\ddot{\theta}$$

$$\therefore \quad a-a_{cm}=R\alpha \quad \text{where } a=\ddot{x}=\text{acceleration of } P$$
$$a_{cm}=\text{acceleration of } O.$$

Considering linear motion of the system,

$$F=(m_1+m_2)a_{cm}. \quad .. \quad \text{(i)}$$

Considering rotational motion of the cylinder,

$$\tau=F.R=(\tfrac{1}{2}m_1R^2)\times\alpha$$

or
$$FR=\left(\frac{1}{2}m_1R^2\right)\times\frac{a-a_{cm}}{R}$$

or
$$F=\tfrac{1}{2}m_1(a-a_{cm}) \quad .. \quad \text{(ii)}$$

Eliminating a_{cm} between (i) and (ii)

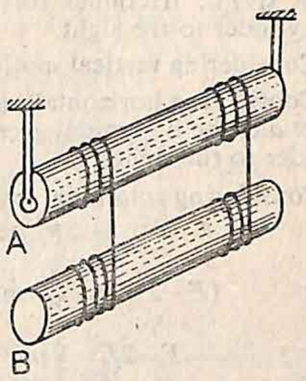
$$a=\frac{F(3m_1+2m_2)}{m_1(m_1+m_2)}.\quad Ans.$$

T (kinetic energy) = work done by external forces

$$=Fs=F(\tfrac{1}{2}at^2) \quad (\because \; s=0+\tfrac{1}{2}at^2)$$

or
$$T=\frac{(3m_1+2m_2)F^2t^2}{2m_1(m_1+m_2)}.\quad Ans.$$

4. The arrangement shown here consists of two identical uniform solid cylinders, each of mas m, on which two light threads are wound symmetrically. Find the tensions of each thread in the process of motion. The friction in the axle of the upper cylinder is assumed to be absent.



Sol. Let the cylinder go down by x when the upper cylinder A rotates through θ and it itself rotates through θ'. Since both cylinders unwind themselves

$$x=R(\theta+\theta'), \quad \text{or} \quad \ddot{x}=R(\ddot{\theta}+\ddot{\theta}'), \quad \text{or } a=R(\alpha+\alpha')$$

where a = downward acceleration of B,

α = angular acceleration of A,

α' = angular acceleration of B.

Considering rotational motion of A, $\tau=2TR=(\frac{1}{2}mR^2)\times\alpha$

or $$\alpha=\frac{4T}{R}.$$

Considering rotational motion of B, $\tau=2T.R=(\frac{1}{2}mR^2)\times\alpha'$

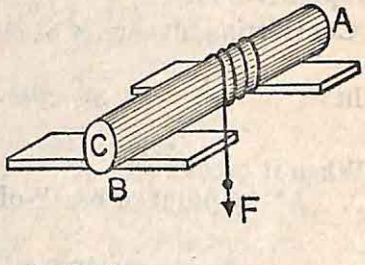
or $$\alpha'=\frac{4T}{R}.$$

Considering downward motion of B, $mg-2T=ma$

or $$mg-2T=mR(\alpha+\alpha')=mR.\left(\frac{4T}{R}+\frac{4T}{R}\right)$$

or $mg-2T=8T$, or $T=\frac{1}{10}mg$. *Ans.*

5. A uniform solid cylinder of mass m rests on two horizontal planks. A thread is wound on the cylinder. The hanging end of the thread is pulled vertically down with a constant force F (see figure). Find the maximum magnitude of the force F which still does not bring about any sliding of the cylinder, if the coefficient of friction between the plank and the cylinder is equal to μ. What is the maximum acceleration of the centre of mass over the planks ?



Sol. The forces acting on the cylinder are : (i) mg, downwards, (ii) F, applied force downwards, (iii) N, reaction of each plank upward, (iv) f_r, frictional force (not limiting) between each plank and the cylinder to the right.

Considering vertical motion, $F+mg=2N$ .. (i).

Considering horizontal motion, $2f_r=ma$ .. (ii)

where a is the horizontal acceleration of the centre of mass of the cylinder to the right.

Considering rotational dynamics of the cylinder about its axis,

$$\tau=F.\,R-2f_rR=(\tfrac{1}{2}\,mR^2)\times\alpha$$

or $$(F-2f_r)R=\frac{1}{2}\,mR^2\times\frac{a}{R} \qquad (\because\ a=\alpha.R)$$

or $$F-2f_r=\tfrac{1}{2}\,ma$$

or $$F=\tfrac{1}{2}\,ma+2f_r=\tfrac{3}{2}\,ma \qquad \text{.. (iii)}$$

$$[\because\ 2f_r=ma \text{ by (ii)}]$$

or $$F=3f_r \qquad \text{.. (iv).}$$

Now, $$f_r\leqslant f_{lim} \quad \text{or} \quad f_r\leqslant\mu N \qquad (\because\ f_{lim}=\mu N)$$

or $$\frac{1}{3}\,F\leqslant\mu\,\frac{F+mg}{2}, \quad \text{or} \quad 2F\leqslant3\mu F+3\mu mg$$

or $$F\leqslant\frac{3\mu mg}{2-3\mu}, \qquad \therefore\ F_{max}=\frac{3\mu mg}{2-3\mu}.\ \textit{Ans.}$$

When the force is maximum, acceleration is automatically maximum.

$$\therefore\ a_{max}=\frac{2F_{max}}{3m}=\frac{2\mu g}{2-3\mu}.\ \textit{Ans.}$$

6. A uniform sphere of radius r rolls without slipping down the top of a sphere of radius R. Find the angular velocity of the sphere at the moment it breaks off the sphere. The initial velocity of the sphere is negligible.

Sol. The centre of mass of the sphere moves in a vertical circle of radius $(R+r)$.

Considering 'dynamics of circular motion' of the centre of mass we have

$$mg\cos\theta-N=\frac{mv^2}{R+r}$$

When it breaks off

$\therefore$ At the point of break-off, $N=0$.

$$mg\cos\theta=\frac{mv^2}{R+r}=\frac{m\omega^2r^2}{R+r} \qquad (\because\ v=\omega r).$$

Considering conservation of energy

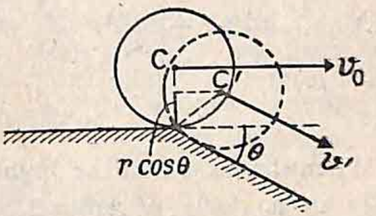
$$mg(R+r)=mg(R+r)\cos\theta+\tfrac{1}{2}\,mv^2+\tfrac{1}{2}\,I\omega^2$$

or $$mg(R+r)=mg(R+r)\cos\theta+\tfrac{1}{2}\,m\omega^2r^2+\tfrac{1}{2}\,(\tfrac{2}{5}\,mr^2)\omega^2$$

or $$mg(R+r)=(R+r).\,\frac{m\omega^2r^2}{(R+r)}+\frac{7}{10}\,m\omega^2r^2$$

or $$g(R+r)=\frac{17}{10}\,\omega^2r^2 \quad \text{or } \omega=\sqrt{\frac{10g(R+r)}{17r^2}}\,. \quad \textit{Ans.}$$

*7. A uniform solid cylinder of radius R rolls over a horizontal plane passing on to an inclined plane forming an angle θ with the horizontal. Find the maximum velocity of the centre of mass v_0 which still permits the cylinder to roll on to the inclined plane *without a jump*. The cylinder rolls without sliding.



Sol. By conservation of energy we have

$$\tfrac{1}{2}\,mv_0^2+\tfrac{1}{2}\,I\omega^2+mgr=\tfrac{1}{2}\,mv'^2+\tfrac{1}{2}\,I\omega'^2+mgr\cos\theta.$$

Since $I=\frac{1}{2}\,mr^2$ and $v=\omega r$ and $v'=\omega' r$ as the cylinder rolls without sliding, $\frac{3}{4}\,mv_0^2+mgr=\frac{3}{4}\,mv'^2+mgr\cos\theta$.

Considering circular motion of the centre of mass of cylinder about O (point of contact), $mg\cos\theta-N=\dfrac{mv'^2}{r}$.

It will not jump so long $N\geqslant 0$ or $mg\cos\theta\geqslant\dfrac{mv'^2}{r}$

or $$g\cos\theta\geqslant\frac{v'^2}{r}$$

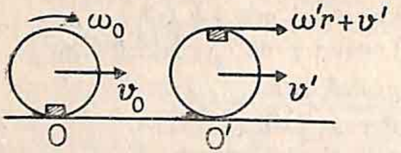
or $$gr\cos\theta\geqslant v_0^2+\tfrac{4}{3}\,gr-\tfrac{4}{3}\,gr\cos\theta$$

or $$\tfrac{7}{3}\,gr\cos\theta-\tfrac{4}{3}\,gr\geqslant v_0^2, \quad \text{or} \quad v_0^2\leqslant\tfrac{1}{3}\,gr(7\cos\theta-4)$$

or $$v_0\leqslant\sqrt{\tfrac{1}{3}\,gr(7\cos\theta-4)}$$

$$\therefore \quad v_{0(max)}=\sqrt{\tfrac{1}{3}\,gr(7\cos\theta-4)}\,. \quad \textit{Ans.}$$

*8. A small mass m is attached to the inside of a rigid ring of the same mass m and radius R. The ring rolls without slipping over a horizontal plane; at the moment when the mass m gets into the lowest position, the centre of the ring moves with velocity v_0. At what value of v_0 will the ring move without bouncing ?



Sol. From the energy conservation between the lowest and highest position of the mass we have

$$\tfrac{1}{2}\, mv_0{}^2 + \tfrac{1}{2}(mR^2)\omega_0{}^2 + mgR + \tfrac{1}{2}\, m(v_0 - \omega_0 R)^2$$
$$= \tfrac{1}{2}\, mv'^2 + \tfrac{1}{2}(mR^2)\omega'^2 + mgR + \tfrac{1}{2}\, m(v' + \omega' R)^2 + mg2R.$$

As the ring rolls without slipping

$$v_0 = \omega_0 R \text{ and } v' = \omega' R$$

$$\therefore \quad \frac{1}{2}\, mv_0{}^2 + \frac{1}{2}\, mR^2 .\ \frac{v_0{}^2}{R^2} = \frac{1}{2}\, mv'^2 + \tfrac{1}{2} mR^2 .\ \frac{v'^2}{R^2} + \tfrac{1}{2}\, m .\ 4v'^2 + 2mgR$$

$$v_0{}^2 = 3v'^2 + 2gR. \qquad \text{.. (i)}$$

When the mass is at the highest position, it moves instantaneously in a circle of radius $2R$ about the point of contact O' as it becomes the instantaneous centre of rotation and the centre of the ring moves in a circle of radius R about O'.

Let N be the reaction of the ring on the mass m and it is downward. Let N' be the reaction of the ground on the ring. Then considering instantaneous circular motion of the mass m and the centre on the ring we have

$$N + mg = m .\ \frac{(2v')^2}{2R} \quad \text{or} \quad N + mg = \frac{2mv'^2}{R} \qquad \text{.. (ii)}$$

and

$$N + N' - mg = m\ \frac{v'^2}{R} \qquad \text{.. (iii).}$$

Subtracting (iii) from (ii) to eliminate N,

$$2mg - N' = \frac{mv'^2}{R}. \qquad \text{.. (iv)}$$

The ring will not bounce so long $N' \geqslant 0$

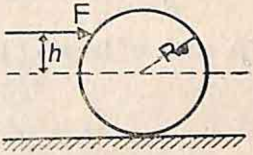
or $2mg \geqslant \dfrac{mv'^2}{R}$ or $v'^2 \leqslant 2gR$ or $3v'^2 \leqslant 6gR$

or $v_0{}^2 - 2gR \leqslant 6gR$ [using (i)]

or $v_0{}^2 \leqslant 8gR$ or $v_0 \leqslant \sqrt{8gR}$

$$\therefore \quad v_{0(max)} = \sqrt{8gR}. \quad \textit{Ans.}$$

*9. A billiard ball, initially at rest, is given a sharp impulse by a cue. The cue is held horizontally a distance h above the central line (see figure). The ball leaves the cue with a speed v_0 and, because of its forward 'rolling + sliding' eventually acquires a final speed of $\frac{9}{7}v_0$. Show that



$h=\frac{4}{5}R$, where R is radius of the ball.

Sol. After the ball leaves the cue, the forces acting on the ball are : mg downward, N reaction upward and frictional force μN in the forward direction. Due to frictional force the linear velocity gradually decreases and the angular velocity increases till the ball starts 'rolling without sliding'. We have from dynamics of linear motion, $\mu mg=ma$, or $a=\mu g$

$$\therefore \quad v=v_0-at=v_0-\mu gt.$$

From consideration of rotational motion, $\tau=\mu mgR=\frac{2}{5}mR^2\times\alpha$

or $$\alpha=\frac{5\mu g}{2R}.$$

From $\omega=\omega_0+\alpha t$, $$\omega=0+\frac{5\mu g}{2R}t$$

or $$\omega=\frac{5\mu g}{2R}t.$$

When the ball starts 'rolling without sliding'

$$v=\omega R, \quad \text{or} \quad v=\tfrac{5}{2}\mu gt.$$

$\therefore$ When the ball rolls without sliding

$$v=v_0-\mu g.\frac{2}{5}\frac{v}{\mu g} \quad \text{or} \quad v=\frac{5}{7}v_0.$$

At this stage (when there is 'rolling without slipping'), the point of contact is the instantaneous centre of rotation. The velocity of a point distance h above the central line is $v+\omega.h$. This is $\frac{9}{7}v_0$.

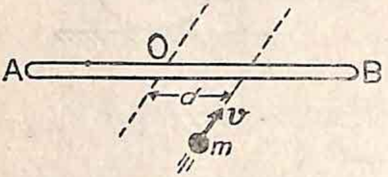
$$\therefore \quad v+\omega h=\tfrac{9}{7}v_0, \quad \text{or} \quad v+\frac{v}{R}h=\frac{9}{7}v_0 \qquad \left(\because \quad \omega=\frac{v}{R}\right)$$

or $$v\left(1+\frac{h}{R}\right)=\frac{9}{7}v_0$$

or $$\frac{5}{7}v_0\left(1+\frac{h}{R}\right)=\frac{9}{7}v_0 \qquad \left(\because \quad v=\frac{5}{7}v_0\right)$$

$\therefore \quad 5+\frac{5h}{R}=9, \quad$ or $\quad h=\frac{4}{5}R.$ **Proved.**

10. A stick of length l lies on horizontal table. It has a mass M and is free to move in any way on the table. A ball of mass m, moving perpendicularly to the stick at a distance d from its centre with speed v collides elastically with it. What quantities are conserved in the collision? What must be the mass of the ball so that it remains at rest immediately after the collision?



Sol. In the collision linear momentum of the system (stick + ball), angular momentum and kinetic energy are conserved.

From conservation of linear momentum, $mv=MV$ .. (i).

From conservation of angular momentum,

$$mvd=I\omega$$

(before collision angular momentum of m is mvd. This is from the fact that moment of momentum is also angular momentum).

From conservation of kinetic energy,

$$\tfrac{1}{2}mv^2=\tfrac{1}{2}I\omega^2+\tfrac{1}{2}MV^2$$

or $\quad mv^2=I.\frac{m^2v^2d^2}{I^2}+M.\frac{m^2v^2}{M^2}, \quad$ or $\quad 1=\frac{md^2}{I}+\frac{m}{M}.$

Now, $\quad I=\frac{1}{12}Ml^2, \quad \therefore \quad 1=\frac{md^2}{\frac{1}{12}Ml^2}+\frac{m}{M}=\frac{12md^2}{Ml^2}+\frac{m}{M}$

or $$m=\frac{Ml^2}{12d^2+l^2}.$$ **Ans.**

11. A planet of mass m moves along an ellipse around the sun (mass m_s) so that its maximum and minimum distances from the sun are equal to r_1 and r_2 respectively. Find the angular momentum L of this planet relative to the centre of the sun.

Sol. Moment of momentum is also angular momentum. At the maximum distances of the planet its velocity is at right angles to the position vectors.

$$\therefore \quad L=mr_1v_1=mr_2v_2 \qquad \text{.. (i).}$$

By the 'energy conservation principle',

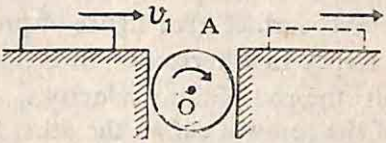
$$-G\frac{mm_s}{r_1}+\frac{1}{2}mv_1^2=-G\frac{mm_s}{r_2}+\frac{1}{2}mv_2^2$$

or $$Gmm_s\left(\frac{1}{r_2}-\frac{1}{r_1}\right)=\frac{1}{2}\,m.\,\frac{L^2}{m^2r_2^2}-\frac{1}{2}\,m.\,\frac{L}{m^2r_1^2}$$

or $$Gmm_s\,\frac{r_1-r_2}{r_1r_2}=\frac{1}{2}\cdot\frac{L^2}{m}\left(\frac{r_1^2-r_2^2}{r_1^2r_2^2}\right)$$

$$L^2=\frac{2Gm^2m_sr_1r_2}{r_1+r_2},\quad\text{or}\quad L=m\sqrt{\frac{2Gm_sr_1r_2}{r_1+r_2}}.\quad \textit{Ans.}$$

12. The axis of a cylinder of radius R and moment of inertia about it axis I is fixed at O. Its highest point A is in level with two plane horizontal surfaces. A block of mass M is initially moving to the right without friction with speed v_1. It passes over the cylinder to the dotted position. Calculate the speed v_2 in the dotted position and the angular velocity acquired by the cylinder.



Sol. Let us consider conservation of angular momentum of the system (cylinder + block) about the axis of the cylinder. We have

$Mv_1R=I\omega+Mv_2R$ (making use of the fact that moment of linear momentum about a point is the angular momentum about that point).

But $$v_2=\omega R$$

$$\therefore\quad Mv_1R=I\frac{v_2}{R}+Mv_2R;\qquad\text{or}\qquad v_2=\frac{v_1}{1+\dfrac{I}{MR^2}}\quad \textit{Ans.}$$

and $$\omega=\frac{v_2}{R}=\frac{v_1}{R\left(1+\dfrac{I}{MR^2}\right)}.\ \textit{Ans.}$$

13. A man of mass m_1 stands on the edge of a horizontal uniform disc of mass m_2 and radius R which is capable of rotating freely about a stationary vertical axis passing through its centre. The man walks along the edge of the disc through angle θ relative to the disc and then stops. Find the angle through which the disc turned by the time the man stopped.

Sol. Just as in linear motion if no external force acts on system, the centre of mass initially at rest remains stationary, so also in rotational motion, if no external torque acts on the system, the centre of rotational inertia initially at rest remains stationary. The centre of inertia is given by similar formula

$$\theta_{ci}=\frac{I_1\theta_1+I_2\theta_2}{I_1+I_2};\qquad cf,\ x_{cm}=\frac{m_1x_1+m_2x_2}{m_1+m_2}.$$

Let θ_1 be the absolute angle of rotation of the man and θ_2 be that of the disc. Then

$$\theta=\theta_1-\theta_2 \qquad \therefore \qquad \theta_1=\theta+\theta_2 \qquad \cdot\cdot \text{ (ii).}$$

Since in the beginning the system is stationary, $\theta_{ct}=0$.

$$\therefore \qquad I_1\theta_1+I_2\theta_2=0; \qquad \text{or} \qquad (m_1R^2)(\theta+\theta_2)+(\tfrac{1}{2}m_2R^2)\theta_2=0.$$

$$2m_1(\theta+\theta_2)+m_2\theta_2=0; \qquad \text{or} \qquad \theta_2=-\frac{2m_1}{2m_1+m_2}\theta \textit{ Ans.}$$

***14** A uniform rod of mass m and length l rests on a smooth horizontal surface (see figure of prob. 10). One of the ends of the rod is struck in a horizontal direction at right angles to the rod. As a result the rod obtains velocity v_0. Find the force with which one half of the rod will act on the other in the process of motion.

Sol. Since moment of momentum is also angular momentum,

L (angular momentum acquired by the rod about its centre) $=I\omega$

$$=mv_0\times\frac{l}{2}$$

$$\text{or} \qquad \left(\frac{m}{12}l^2\right)\omega=mv_0\,l/2 \qquad \text{or} \qquad \omega=\frac{6v_0}{l}.$$

The two halves of the rod will exert on each other the centripetal force needed for their circular motion. Consider an element at a distance x and of thickness dx. Mass of the element $=\left(\frac{m}{l}dx\right)$.

$$\therefore \quad \text{Centripetal force on the element}=\left(\frac{m}{l}dx\right)\omega^2x.$$

$$\therefore \quad \text{Total centripetal force on one half}=\int_0^{\frac{1}{2}}\frac{m}{l}\omega^2x\,dx$$

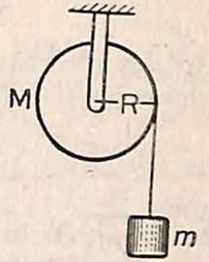
$$=\frac{m\omega^2}{l}\cdot\frac{l^2}{8}=\frac{1}{8}m\omega^2l$$

or F (force exerted by one half on the other) $=\frac{1}{8}\,ml\,\frac{36v_0^2}{l^2}$

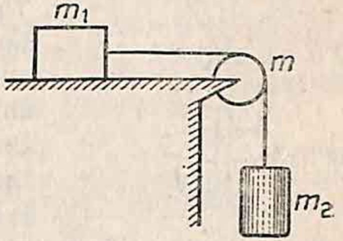
$$=\frac{9}{2}\frac{mv_0^2}{l}.\textit{ Ans.}$$

EXERCISES

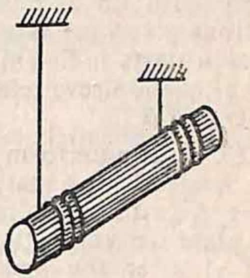
1. A light thread with a body of mass m tied to its end is wound on uniform solid cylinder of mass M and radius R. At the moment $t=0$ the system is set in motion. Calculate the angular velocity of the cylinder at time t and the kinetic energy of the whole system at that time.



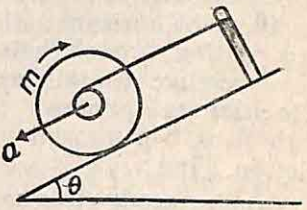
2. In the system shown in the figure, the masses of the bodies are m_1 and m_2, the coefficient of friction between the body m_1 and the horizontal plane is equal to μ, and a frictionless pulley of mass m is assumed to be a uniform disc. At the moment $t=0$ the body m_2 starts descending. Find the work done by the frictional forces acting on body m_1 over the first t seconds after the beginning of the motion.



3. A uniform cylinder of mass $m = 8$ kg and radius $R = 1{\cdot}3$ cm starts descending at the moment $t=0$ due to gravity. Find the tension in each thread and the angular acceleration of the cylinder. Also find the instantaneous power developed by the gravitational force.



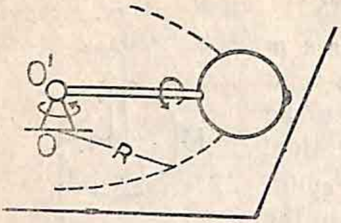
4. A spool of thread of mass m is placed on an inclined smooth plane set at an angle θ to the horizontal. The free end of the thread is attached to the wall as shown in the figure. Calculate the acceleration of the centre of mass of the spool, if its moment of inertia about its axis is I and the radius of the wound thread layer is r.



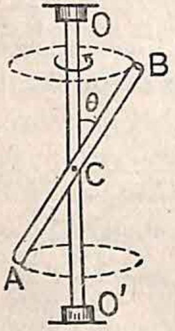
***5.** A uniform solid cylinder of mass m and radius R is set in rotation about its axis with an angular velocity ω_0, then lowered onto a horizontal surface and released. The coefficient of friction between the cylinder and the plane is equal to μ. Find :

(a) how long the cylinder will move *with* sliding,

(b) the total work done by the sliding friction force acting on the cylinder.



6. A uniform sphere of mass m and radius r rolls without sliding over a horizontal plane, rotating a horizontal axle. In the process the centre of mass of the sphere moves with velocity v along a circle of radius R. Find the kinetic energy of the sphere.



*7. A thin uniform rod AB of mass m and length l is rigidly attached at its midpoint to a rigid rotation axis OO'. The axis is set into rotation with a constant angular velocity ω. Find the resultant moment of the centrifugal force about the point C where the rod is attached to the axis. The inclination of the rod AB to the axis of rotation OO' is θ.

8. A smooth uniform rod AB of mass M and length l rotates freely with an angular velocity ω_0 in a horizontal plane about a stationary vertical axis passing through its end A. A small sleeve of mass m starts sliding along the rod from the point A. Find the velocity v' of the sleeve relative to the rod at the moment it reaches the other end B.

*9. A thin uniform square plate with side l and mass M can rotate freely about a stationary vertical axis coinciding with one of its sides. A small ball of mass m flying with velocity v at right angles to the plate strikes elastically the centre of it. Find the velocity of the ball v' after the impact and the horizontal component of the force which the axis will exert on the plate after the impact.

10. Two horizontal discs rotate freely about a common vertical axis passing through their centres. The moments of inertia of the discs relative to this axis are equal to I_1 and I_2, and their angular velocities are ω_1 and ω_2. When the upper disc fell on the lower one, both discs, began rotating, after some time, as a single body due to friction. Find :

(a) the steady state angular velocity of the discs;
(b) the work performed by the frictional forces in the process.

11. The moon (mass m) revolves about the earth (mass M) so we always see the same face of the moon. (a) How are the spin and orbital angular momentum of the moon (with respect to earth) related ? (b) By how much will its spin angular momentum have to change if we were to be able to see all the moon's surface during the course of a month ? Take r_m = radius of the moon and d = distance of the moon from the centre of the earth.

TEST YOURSELF

1. Answer the following questions— $5 \times 3 = 15$

(i) Consider a small merry-go-round of radius 4 m and mass 10 kg. The moment of inertia about its axis of rotation (mass concentrated on the rim) is

(a) 40 kgm^2, (b) 160 kgm^2, (c) 80 kgm^2, (d) 60 kgm^2.

(ii) Two identical dumbbell of length l are fixed at their centres at inclination θ. Each dumbbell consists of two masses, each mass equals to m. The moment of inertia of the system is

(a) $ml^2\cos\theta$, (b) $2ml^2$, (c) $4\,ml^2\sin\theta$, (d) ml^2.

(iii) A hollow sphere, a ring, a disc and a solid sphere, all of the same mass and radius, are allowed to roll without slipping down an inclined plane. The one which reaches the bottom first is (a) the hollow sphere, (b) the ring, (c) the disc, (d) the solid sphere.

(iv) In the above example the one which will have the greatest speed on reaching the bottom of the plane (all cover the same distance down the plane) is (a) the hollow sphere, (b) the ring, (c) the disc, (d) the solid sphere.

(v) The dimension of moment of inertia is

(a) MLT, (b) ML^2, (c) $M^2L^2T^{-1}$, (d) MLT^{-1}.

2. Answer the following questions— $5 \times 3 = 15$

(i) Is angular momentum of a system conserved always ? If not, under what conditions ?

(ii) If the atmosphere suddenly condenses into a solid mass as thin layer on the surface of the earth, what effect would be on the earth's time of rotation about its axis ?

(iii) A metre stick, half of which wood—the other half steel—is pivoted at the wooden end and a force is applied to the steel end. Next it is pivoted at the steel end and the same force is applied at the wooden end. Does one get the same angular acceleration in each case ? Explain.

(iv) A spinning top stands erect, but the one not spinning falls. Why ?

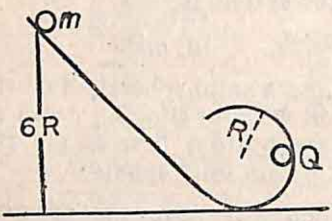
(v) When a wheel rolls without slipping, if v_{cm} is the speed of the centre of the wheel, the velocity of the highest point is $2\,v_{cm}$. Is it true ? Explain.

Answer any five questions— $5 \times 20 = 100$

1. (a) A body of radius R and mass m is rolling horizontally without slipping with speed v. It then rolls up a hill to a maximum

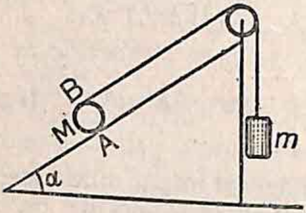
height h. If $h=\frac{3}{4}\frac{v^2}{g}$, what might the body be ? What is the body's rotational inertia ?

(b) A sphere rolls up an inclined plane of inclination angle 30°. At the bottom of the incline the centre of mass of the sphere has translational speed of 5 m/s. How high does the sphere travel up the plane ?



2. (a) A small solid marble of mass m and radius r rolls without slipping along the loop-the-loop as shown in the figure. If it starts from rest at a height of $6R$ above the bottom (R=radius of the circular track) what is the force (horizontal and vertical components) acting on it at the point Q ?

(b) A 10 m long ladder rests against a wall and makes an angle of 60° with horizontal floor. If it starts to slip, where is the instantaneous axis of rotation ?



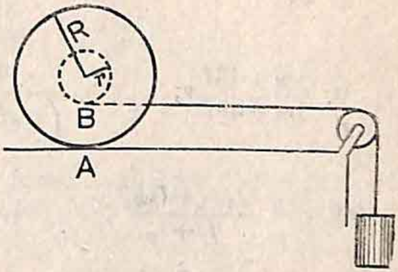
3. (a) A cylinder of mass 30 kg and radius 10 cm has a light thin tape wound around it. The tape passes over a light, smooth fixed pully to a 6 kg body. Find the tension in the tape and the linear acceleration of the cylinder up the incline, assuming no slip.

(b) A metre stick is held vertically with one end on the floor and is then allowed to fall. Find the speed of the other end when it hits the floor, assuming that the end on the floor does not slip.

4. (a) A billiard ball is struck by a cue. The line of action of the applied impulse is horizontal and passes through the centre of the ball. The initial velocity of the ball is v_0. If r is the radius, M is the mass and μ is the coefficient of friction between the ball and the floor, find how far the ball moves before it ceases to slip on the floor.

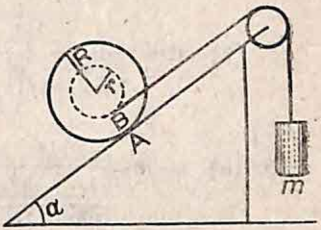
(b) A hoop of radius r rotating with an angular velocity ω_0 is placed on a rough horizontal surface. Find the velocity of the centre of the hoop when it ceases to slip. At the initial moment the velocity of the centre of the hoop was zero.

5. (a) A spool (consider it as a double disc system joined by a short tube at their centre) is placed on horizontal surface. A light string wound several times over the short connecting tube leaves it tangentially and passes over light pulley. A weight of mass m is attached to the end of the string. The radius of the connecting tube is r and mass of the spool is M and radius is R. Find the acceleration of the falling mass m. Neglect the mass of the connecting tube and slipping of the spool.



(b) In a playground there is a small merry-go-round of radius 2 m and mass 10 kg. The radius of gyration is 1·5 m. A child of mass 30 kg runs at a speed of 5 m/s tangent to the rim of the merry-go-round when it is at rest and then jumps on. Neglect friction and find the angular velocity of the merry-go-round and child.

6. (a) A spool (a double disc system connected by a short tube at the centre) of mass M and radius R lies on inclined plane. A light thread is wound on the connecting tube of the spool and its free end carrying a weight having mass m is thrown over a weightless pulley. Determine the angle of inclination 'α' of the plane to the horizontal at which the centre of gravity of the spool will be at rest.



(b) Calculate the kinetic energy of a tractor crawler belt of mass m if the tractor moves with velocity v. There is no slipping.

Full marks 100 Your score

Answers to Exercises

1. $\omega=\dfrac{2mgt}{(2m+M)R}$; $T=\dfrac{m^2g^2t^2}{2m+M}$. 2. $W_f=-\frac{1}{2}\dfrac{\mu(m_2-\mu m_1)m_1g^2t^2}{(m_1+m_2+\frac{1}{2}m)}$.

3. $\frac{2}{3}g$; $\frac{1}{6}mg$; $\frac{2}{3}mg^2t$. 4. $a=\dfrac{g\sin\theta}{1+\dfrac{I}{mr^2}}$.

5. $t=\dfrac{\omega_0 R}{3\mu g}$; $W_{friction}=-\dfrac{1}{6}m\omega_0{}^2R^2$. 6. $T=\dfrac{7}{10}mv^2\left(1+\dfrac{2}{7}\dfrac{r^2}{R^2}\right)$.

7. $\frac{1}{24}m\omega^2l^2\sin 2\theta$. 8. $v' = \frac{\omega_0 l}{\sqrt{1+\frac{3m}{M}}}$.

9. $\frac{3m-4M}{3m+4M}v$; $\frac{4v}{\left(1+\frac{4M}{3m}\right)l}$.

10. $\omega = \frac{I_1\omega_1+I_2\omega_2}{I_1+I_2}$, $W_{friction} = -\frac{1}{2}\frac{I_1I_2}{I_1+I_2}(\omega_1-\omega_2)^2$.

11. $\frac{2}{5}\left(\frac{r_m}{d}\right)^2$.

Answers of Test Questions

1. (a) $\frac{R}{\sqrt{2}}$, (b) 3·6 m. 2. (a) $\frac{2}{7}$ mg, $\frac{50}{7}$ mg, (b) 5 m, 5/3 m.

3. (a) 3·4 m/s²; (b) 5·4 m/s. 4. (a) $\frac{12}{49}$ $\frac{v_0^2}{\mu g}$, (b) $\frac{1}{2}\omega_0 r$.

5. (a) $a = \frac{2gh}{2m+3M\left(\frac{R}{R-r}\right)^2}$; (b) 7·27 rad/s.

6. (a) $\sin\alpha = \frac{1}{\frac{r^2}{R^2}+\frac{M}{2m}}$, (b) mv^2.

Solution to Exercises

1. Considering linear motion of m we have

$$mg - T = ma \quad .. \quad \text{(i)}.$$

Considering angular motion of the cylinder

$$\tau = T\times R = (\tfrac{1}{2}MR^2)\times\alpha$$

or

$$T = \tfrac{1}{2}MR\alpha. \quad .. \quad \text{(ii)}.$$

Adding (i) and (ii,) $mg = ma + \frac{1}{2}MR\alpha$.

But $a = \alpha R$.

$$\therefore \quad a = \frac{2mg}{2m+M} \text{ and } \alpha = \frac{2mg}{(2m+M)R}.$$

From $\omega = \omega_0 + \alpha t$, $\omega = 0 + \frac{2mgt}{(2m+M)R}$

or

$$\omega = \frac{2mgt}{(2m+M)R}. \ \textit{Ans.}$$

$$T=\tfrac{1}{2}mv^2+\tfrac{1}{2}I\omega^2=\tfrac{1}{2}m\omega^2R^2+\tfrac{1}{2}(\tfrac{1}{2}MR^2)\,\omega^2 \quad (\because \ v=\omega R)$$
$$=\tfrac{1}{4}\omega^2R^2(2m+M)$$
$$=\frac{1}{4}\cdot\frac{4m^2g^2t^2R^2}{(2m+M)^2R^2}(2m+M)$$

or $$T=\frac{m^2g^2t^2}{(2m+M)}.\quad \textit{Ans.}$$

2. Since the mass of the pulley is not negligible, the tensions in the horizontal and vertical portions of the string are different. Let them be T_1 and T_2.

Considering motion of m_1, $\quad T_1-\mu m_1g=m_1a \quad ..\quad$ (i)
,, ,, ,, m_2, $\quad m_2g-T_2=m_2\alpha \quad ..\quad$ (ii)
,, ,, ,, pulley, $\quad \tau=T_2r-T_1r=(\tfrac{1}{2}mr^2)\times\alpha$
or $\quad T_2-T_1=\tfrac{1}{2}mr\times a/r \quad (\because \ a=\alpha r)$
or $\quad T_2-T_1=\tfrac{1}{2}ma \quad ..\quad$ (iii).

Solving (i), (ii) and (iii) for α

$$\alpha=\frac{(m_2-\mu m_1)g}{(m_1+m_2+\frac{1}{2}m)}.$$

By 'work-energy' theorem

$$T_1s+W_{friction}=\tfrac{1}{2}m_1v^2$$

or $$W_{friction}=\tfrac{1}{2}m_1\,(2as)-T_1s=m_1as-(\mu m_1g+m_1a)s$$
$$=-\mu m_1gs$$

or $$W_{friction}=-\mu m_1g(0+\tfrac{1}{2}.at^2)=-\tfrac{1}{2}\mu m_1g.\frac{(m_2-\mu m_1)gt^2}{(m_1+m_2+\frac{1}{2}m)}$$

or $$W_{friction}=-\tfrac{1}{2}\frac{\mu(m_2-\mu m_1)m_1g^2t^2}{(m_1+m_2+\frac{1}{2}m)}.\quad \textit{Ans.}$$

3. Considering linear motion of the cylinder,

$$mg-2T=ma \quad ..\quad \text{(i)}.$$

Considering rotational motion of the cylinder,

$$\tau=2T.R=(\tfrac{1}{2}\,mR^2)\times\alpha.$$

But $\quad a=\alpha R, \quad \therefore \quad 2T.R=(\tfrac{1}{2}\,mR^2)\times a/R$
or $\quad 4T=ma \quad ..\quad$ (ii).
Solving (i) and (ii), $\quad a=\tfrac{2}{3}\,g. \quad$ *Ans.*
$\therefore \quad T=\tfrac{1}{6}\,mg. \quad$ *Ans.*

$$P\,(\text{power})=\text{force}\times\text{velocity}=mgv=mg\omega R=mgR\left(\frac{2}{3}\,\frac{g}{R}t\right)$$

$$\left(\because \quad \omega=0+\alpha t=\frac{2}{3}\,\frac{g}{R}\,t\right)$$

or $$P=\tfrac{2}{3}\,mg^2t. \quad \textit{Ans.}$$

4. Here the forces acting on the spool are : (i) mg downwards, (ii) T, tension of thread up the plane. There is no friction as the plane is smooth. Let the centre go down the plane by x when the spool rotates through θ. Then

$$x=r\theta \quad \text{or} \quad \ddot{x}=r\,\ddot{\theta}$$

or

$$a=r\alpha$$

where a=acceleration of the center of mass of the spool down the plane and α is the angular acceleration of it.

Considering its rotational motion, $\tau=Tr=I\alpha$

or

$$Tr=I\,\frac{a}{r} \qquad \text{.. (i).}$$

Considering its linear motion down the plane

$$mg\sin\theta-T=ma, \quad \text{or} \quad mg\sin\theta-\frac{Ia}{r^2}=ma$$

or

$$a=\frac{g\sin\theta}{1+\dfrac{I}{mr^2}}. \quad \textit{Ans.}$$

5. The forces acting on the cylinder are : (i) mg, downwards, (ii) N, reaction upward, (iii) μN, sliding frictional force.

Considering linear motion, $\mu N=ma$ where a is the forward acceleration produced by friction,

and $N=mg$; $\therefore$ $\mu mg=ma$, or $a=\mu g$ .. (i).

Considering rotational motion about axis of the cylinder,

$$\tau=\mu mgR=(\tfrac{1}{2}\,mR^2)\times\alpha$$

or

$$\alpha=\frac{2\mu g}{R} \qquad \text{.. (ii).}$$

So long it slides $a\neq\alpha R$. As the cylinder slides, its linear velocity increases and angular velocity decreases till v becomes equal to ωR. Let t be the time in which v becomes equal to ωR.

From $\omega=\omega_0+\alpha t$, $\quad \omega=\omega_0-\dfrac{2\mu g}{R}\,t$ .. (iii).

From $v=v_0+at$, $\quad v=0+\mu gt$ .. (iv).

Now $\quad v=\omega R$

$$\therefore \quad \mu gt=\left(\omega_0-\frac{2\mu gt}{R}\right)R$$

or $\mu gt=\omega_0R-2\mu gt$, or $t=\dfrac{\omega_0R}{3\mu g}$. *Ans.*

By work-energy theorem

$$W_{friction}=(\tfrac{1}{2}\,mv^2+\tfrac{1}{2}\,I\omega^2)-\tfrac{1}{2}\,I\omega_0^2 \qquad \text{(work done by } mg \text{ and } N \text{ is zero)}$$

$$=\tfrac{1}{2}\,m\omega^2R^2+\tfrac{1}{2}.\tfrac{1}{2}\,mR^2\omega^2-\tfrac{1}{2}\,I\omega_0^2$$

$$=\tfrac{3}{4}\,m\omega^2R^2-\tfrac{1}{2}\,(\tfrac{1}{2}\,mR^2\omega_0^2)=\tfrac{3}{4}\,m\omega^2R^2-\tfrac{1}{4}\,mR^2\omega_0^2.$$

Now, from (iii), $\omega=\omega_0-\dfrac{2\mu g}{R}\cdot\dfrac{v}{\mu g} \qquad \left(\because\ t=\dfrac{v}{\mu g}\right)$

$$=\omega_0-\frac{2}{R}.\,\omega R \qquad (v=\omega R)$$

$$3\omega=\omega_0, \qquad \omega=\tfrac{1}{3}\,\omega_0.$$

$$\therefore\quad W_{friction}=\frac{3}{4}.\,m.\,\frac{\omega_0^2}{9}\,R^2-\frac{1}{4}\,mR^2\omega_0^2=-\frac{1}{6}\,m\omega_0^2R^2.\quad Ans.$$

6. The sphere possesses two rotational kinetic energies : one about the diameter (OA) and it is $\tfrac{1}{2}\,I\omega^2$ and the other due to rotation about the axis OO' and it is $\tfrac{1}{2}\,I'\omega'^2$ where I' is the moment of inertia about the axis OO' and ω' is the angular velocity of its centre of mass about OO'.

$$\therefore\quad T=\tfrac{1}{2}\,I\omega^2+\tfrac{1}{2}\,I'\omega'^2.$$

Now $I=\tfrac{2}{5}\,mr^2$ and $I'=(\tfrac{2}{5}\,mr^2+mR^2)$ by theorem of parallel axes.

Since the sphere rolls without slipping, $v=\omega r.$

Also $v=\omega' R.$

$$\therefore\quad T=\frac{1}{2}\left(\frac{2}{5}\,mr^2\right)\cdot\frac{v^2}{r^2}+\frac{1}{2}\left(\frac{2}{5}\,mr^2+mR^2\right)\cdot\frac{v^2}{R^2}$$

$$=\tfrac{1}{5}\,mv^2+\tfrac{1}{5}\,mv^2\,(r^2/R^2)+\tfrac{1}{2}\,mv^2$$

$$=\tfrac{7}{10}\,mv^2+\tfrac{1}{5}\,mv^2\,(r^2/R^2)$$

or $$T=\frac{7}{10}mv^2\left(1+\frac{2}{7}\,\frac{r^2}{R^2}\right).\qquad Ans.$$

7. Consider an element of length dx at a distance x from C. The mass of the element is $\left(\dfrac{m}{l}\right)dx$. It rotates in a circle of radius $x\,\sin\theta$.

$\therefore$ Centrifugal force on it $=\left(\dfrac{m}{l}\,dx\right)\omega^2x\,\sin\theta.$

Moment of this force about $C=$ force $\times$ perp. distance

$$=\left(\frac{m}{l}\,dx\omega^2x\,\sin\theta\right)\times x\cos\theta=\frac{m\omega^2}{2l}\,\sin 2\theta\;x^2dx.$$

$\therefore$ Total centrifugal torque is given by

$$\tau_{centrifugal}=\frac{m\omega^2}{2l}\sin 2\theta\int_{-l/2}^{+l/2}x^2dx$$

$$=\tfrac{1}{24}m\omega^2l^2\sin 2\theta. \qquad \textit{Ans.}$$

8. Considering conservation of angular momentum

$$I\omega_0=(I+ml^2)\omega \qquad \text{.. (i).}$$

Considering conservation principle of energy

$$\tfrac{1}{2}I\omega_0^{\,2}=\tfrac{1}{2}(I+ml^2)\omega^2+\tfrac{1}{2}mv'^2$$

or $$I\omega_0^{\,2}=(I+ml^2)\frac{I^2\omega_0^{\,2}}{(I+ml^2)^2}+mv'^2$$

$$I\omega_0^{\,2}(I+ml^2)=I^2\omega_0^{\,2}+mv'^2(I+ml^2)$$

or $$I\omega_0^{\,2}ml^2=mv'^2(I+ml^2).$$

Putting $I=\tfrac{1}{3}Ml^2$

$$\tfrac{1}{3}Ml^2\omega_0^{\,2}l^2=v'^2(\tfrac{1}{3}Ml^2+ml^2)$$

or $$\frac{1}{3}M\omega_0^{\,2}l^2=v'^2\left(m+\frac{M}{3}\right)$$

or $$M\omega_0^{\,2}l^2=v'^2(3m+M)$$

$$v'=\frac{\omega_0 l}{\sqrt{1+\dfrac{3m}{M}}}. \qquad \textit{Ans.}$$

9. Considering conservation of angular momentum of the system (plate + ball)

$$mv\times l/2=mv'\times(l/2)+I\omega \qquad \text{.. (i).}$$

Applying 'kinetic energy conservation principle in elastic collision',

$$\tfrac{1}{2}mv^2=\tfrac{1}{2}mv'^2+\tfrac{1}{2}I\omega^2$$

or $$mv^2=mv'^2+I\omega^2 \qquad \text{.. (ii).}$$

Eliminating ω between (i) and (ii)

$$\frac{m^2l^2(v-v')^2}{4m(v^2-v'^2)}=\frac{I^2\omega^2}{I\omega^2}$$

or $$\frac{ml^2(v-v')}{4(v+v')}=I=\frac{1}{3}Ml^2 \qquad \left(\because\quad I=\frac{1}{3}Ml^2\right)$$

or $$3m(v-v')=4M(v+v')$$

or $$v'=\frac{3m-4M}{3m+4M}\times v. \qquad \textit{Ans.}$$

Substituting v' in (i) we have

$$\omega=\frac{12mv}{(3m+4M)l}=\frac{4v}{\left(1+\frac{4M}{3m}\right)l}. \quad \textit{Ans.}$$

After the impact the plate rotates about the axis, which exerts the necessary centripetal force.

$\therefore$ F (force of the axis on the plate) = centripetal force = $M\omega^2 l/2$

or
$$F=Ml/2\,.\,\frac{16v^2}{\left(1+\frac{4M}{3m}\right)^2 l^2}$$

or
$$F=\frac{8Mv^2}{l\left(1+\frac{4M}{3m}\right)^2}. \quad \textit{Ans.}$$

10. By conservation angular momentum

$$I_1\omega_1+I_2\omega_2=(I_1+I_2)\omega$$

or
$$\omega=\frac{I_1\omega_1+I_2\omega_2}{I_1+I_2}. \quad \textit{Ans.}$$

By the work-energy theorem

$$W_{friction}=\text{change in kinetic energy}=T_f-T_i$$

or
$$W_{fr}=\tfrac{1}{2}(I_1+I_2)\omega^2-(\tfrac{1}{2}I_1\omega_1^2+\tfrac{1}{2}I_2\omega_2^2)$$

$$=\frac{1}{2}(I_1+I_2)\,\frac{(I_1\omega_1+I_2\omega_2)^2}{(I_1+I_2)^2}-\frac{1}{2}(I_1\omega_1^2+I_2\omega_2^2)$$

$$=\frac{1}{2}\,\frac{(I_1\omega_1+I_2\omega_2)^2}{I_1+I_2}-\frac{1}{2}(I_1\omega_1^2+I_2\omega_2^2)$$

$$W_{fr}=-\frac{I_1I_2}{2(I_1+I_2)}(\omega_1-\omega_2)^2. \quad \textit{Ans.}$$

11. Since we always see the same face of the moon its period for 1 revolution about its axis must be exactly equal to its period of revolution round the earth. That is.

$$T_{spin}=T_{orbit} \quad \text{or} \quad \omega_{spin}=\omega_{orbit}.$$

Now,
$$L_{spin}=I\omega_{spin}=\tfrac{2}{5}\,mr_m^2\times\omega_{spin}$$
$$L_{orbital}=I_{orbit}\,\omega_{orbit}=md^2\times\omega_{orbit}$$ (here the moon is treated as a particle)

$$\therefore \quad \frac{L_{spin}}{L_{orbital}}=\frac{\frac{2}{5}r_m^2}{d^2}=\frac{2}{5}\left(\frac{r_m}{d}\right)^2. \quad \textit{Ans.}$$

Obviously if ω_{spin} is increased to $2\omega_{spin}$ or reduced to $\frac{1}{2}\,\omega_{spin}$, we will see all the moon's surface during the course of a month.

Solutions to Test Questions

1. (a) f (retardation up the plane) $= \dfrac{g \sin\alpha}{\left(1+\dfrac{k^2}{K^2}\right)}$.

From $v^2 - v_0^2 = 2fs$, $\quad v^2 = 2 \times \dfrac{g \sin\alpha}{1+\dfrac{k^2}{R^2}} \times \dfrac{h}{\sin\alpha} \qquad (\because \; h = s \sin\alpha)$

or $$v^2 = \frac{2gh}{1+\dfrac{k^2}{R^2}} = \frac{2g}{1+\dfrac{k^2}{R^2}} \times \frac{3}{4}\,\frac{v^2}{g}$$

or $$k = \frac{R}{\sqrt{2}}. \qquad \textit{Ans.}$$

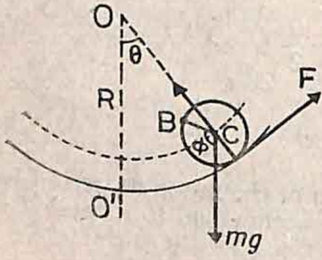
This is the radius of gyration of a disc. Hence the body is a disc.

(b) f (retardation up the plane) $= \dfrac{g \sin 30°}{1+\frac{2}{5}} = \dfrac{5}{7}\, g \times \dfrac{1}{2} = \dfrac{5}{14}\, g$

$(\because \; k^2 = \frac{2}{5} r^2$ for a sphere).

From $\quad v^2 - v_0^2 = 2fs,, \quad 5^2 = 2 \times \frac{5}{14}\, g \times s$

or $$s = \frac{35}{9{\cdot}8} - 3{\cdot}6 \text{ m.} \qquad \textbf{Ans.}$$



2. (a) The centre of the small marble will obviously move in a circular path of radius $(R-r)$.

Let θ be the angle made by the radius vector at time t with the vertical OO'. Then $\dfrac{d\theta}{dt} = \dot{\theta} =$ angular velocity of the centre of the marble on the circular path.

$\therefore \quad a_n$ (normal acceleration)

$= (R-r)\omega^2 = (R-r)\dot{\theta}^2$

s (distance described by the marble) $= (R-r)\theta$

$$\therefore \quad \frac{ds}{dt} = (R-r)\dot{\theta} \qquad \text{or} \qquad v = (R-r)\dot{\theta}$$

and a_t (tangential acceleration) $= \dfrac{dv}{dt} = (R-r)\,\ddot{\theta}$.

Since there is no *slipping* we must take the arc $O'A$ is equal to arc BA.

$$\therefore \quad r\ \theta+\varphi)=\omega\theta \qquad \text{or} \qquad \varphi=\frac{R-r}{r}\,\theta.$$

Considering tangential motion, $F-mg\ \sin\theta=ma_t=m(R-r)\theta$

and considering normal motion, $N-mg\ \ \cos\theta=ma_n=m(R-r)\dot{\theta}^2$

Considering rotational motion, $(mk^2)\ddot{\varphi}=-F.r$

or $\quad (mk^2)\,\dfrac{R-r}{r}\,\ddot{\theta}=-F.\,r \qquad$ (from $\tau=I\alpha$ formula)

or $\quad (mk^2)\,\dfrac{R-r}{r}\left\{\dfrac{F-mg\sin\theta}{m(R-r)}\right\}=-Fr$

or $\quad F=\dfrac{mg\ \sin\theta}{1+(r^2/k^2)}$

When $\theta=\pi/2$, and $k^2=2/5\ r^2$ (for a sphere),

$$\therefore \quad F=\frac{mg}{1+\frac{5}{2}}=\frac{2}{7}\ mg. \quad \textit{Ans.}$$

$N=\dfrac{mv^2}{R-r}$. $\left(\text{by the formula centripetal force}=\dfrac{mv^2}{r}\right)$.

Considering 'energy conservation'

$$mg6R=\frac{1}{2}\ mv^2+\frac{1}{2}\ I\omega^2+mgR=\frac{1}{2}\ mv^2+\frac{1}{2}.\ \ \tfrac{2}{5}mr^2.\ \frac{v^2}{r^2}+mgR$$

or $\quad 5mgR=\frac{7}{10}\ mv^2, \qquad$ or $\qquad mv^2=\frac{50}{7}\ mgR$

$\therefore \quad N=\dfrac{\frac{50}{7}\ mgR}{R-r}=\frac{50}{7}\ mg \qquad (\because \quad R\geqslant r)$

or $\quad N=\frac{50}{7}\ mg. \quad$ *Ans.*

(b) When the rod begins to slip its positions change from instant to instant. At any instant two successive positions (one a little before that instant and the other a little later) can be represented by rotation of the rod about a certain point called 'instantaneous centre'. At any position of the rod the instantaneous centre is the point of intersection of the vertical line through the lower end and the horizontal line through the upper end. Therefore, coordinates of the instantaneous centre of rotation are 10 cos 60° and 10 sin 60° or 5 *m* and 5/3 *m*. *Ans.*

3. (a) Since the cylinder does not slip, the displacement, velocity

and acceleration of the hanging mass are respectively equal to the displacement, velocity and acceleration of the upper point B of the cylinder as the string leaves tangentially the upper point.

Now v (velocity of the upper point) $= v_{cm} + \omega r$

($\because$ velocity of any point of a rotating + sliding body is equal to the vector sum of velocity of the centre of mass plus the velocity due to rotational motion)

or $v = v_{cm} + v_{cm}$ ($\because$ $\omega r = v_{cm}$ when there is no sliding)

$v = 2v_{cm}$.

From 'conservation of energy' when m lowers through h

$$mgh = \tfrac{1}{2} mv^2 + (\tfrac{1}{2} Mv^2_{cm} + \tfrac{1}{2} I\omega^2) \quad (\because \text{ loss in PE} = \text{gain in KE})$$

$$= \frac{1}{2} mv^2 + \frac{1}{2} M \frac{v^2}{4} + \frac{1}{2}\left(\frac{1}{2} Mr^2\right) . \frac{v^2_{cm}}{r^2} \left(\because\ I = \frac{1}{2} Mr^2\right)$$

$$= \frac{1}{2} mv^2 + \frac{1}{8} Mv^2 + \frac{1}{4} M \frac{v^2}{4} = \frac{1}{2} mv^2 + \frac{3}{16} Mv^2$$

or $16\, mgh = 8mv^2 + 3Mv^2$

$$\therefore \quad v^2 = \frac{16mgh}{8m + 3M}$$

From $v^2 = 2fs$, $v^2 = 2ah$

or $$a = \frac{8mg}{8m + 3M} = \frac{g}{1 + \dfrac{3M}{8m}}$$

or $$a = \frac{9{\cdot}8}{1 + \dfrac{3 \times 30}{8 \times 6}} = \frac{9{\cdot}8 \times 8}{23} = 3{\cdot}4 \text{ m/s}^2. \qquad \textit{Ans.}$$

Considering downward motion of m,

$$mg - T = ma$$

or $T = mg - ma = 6(9{\cdot}8 - 3{\cdot}4) = 6 \times 6{\cdot}4 = 38{\cdot}4$ N. *Ans.*

(b) From conservation of energy (loss in PE = gain in KE)

$$mg\, l/2 = \tfrac{1}{2} I\omega^2 = \tfrac{1}{2}\, (\tfrac{1}{3}\, ml^2)\omega^2$$

$$\omega = \sqrt{\frac{3g}{l}}$$

$\therefore$ $v = \omega l = \sqrt{3gl} = \sqrt{3 \times 9{\cdot}8 \times 1} = 5{\cdot}4$ m/s. *Ans.*

4. (a) Here due to friction linear velocity decreases and due to 'torque of frictional force' angular velocity increases.

$$\mu mg = ma \quad \text{or} \quad a = \mu g.$$

$$\therefore \quad v^2 - v_0^2 = -2as$$

or $$v_0^2 - v^2 = 2\mu g s \qquad .. \quad \text{(i)}$$

$$\tau = \mu m g r = I\alpha = (\tfrac{2}{5}\, m r^2)\alpha$$

$$\alpha = \frac{5\mu g}{2r}.$$

$$\therefore \quad \omega = 0 + \alpha t = \frac{5\mu g}{2r}\, t$$

and $$v = v_0 - a t = v_0 - \mu g t.$$

When it begins to roll without slip

$$v = \omega r$$

$$\therefore \quad v/r = \frac{5\mu g}{2r}\, t, \quad \text{or} \quad t = \frac{2v}{5\mu g}$$

$$\therefore \quad v = v_0 - \mu g.\, \frac{2v}{5\mu g} = v_0 - \frac{2v}{5}$$

or $$v = \tfrac{5}{7}\, v_0 \qquad .. \quad \text{(ii)}.$$

$\therefore$ From (i) $$v_0^2 - \tfrac{25}{49}\, v_0^2 = 2\mu g s$$

or $$s = \frac{12}{49}\frac{v_0^2}{\mu g}. \qquad \textit{Ans.}$$

(b) Here due to friction linear velocity increases and angular velocity decreases due to torque of frictional force.

$$\mu m g = m a \quad \text{or} \quad a = \mu g$$

$$\therefore \quad v = 0 + \mu g t \text{ or } \quad v = \mu g t \qquad .. \quad \text{(i)}$$

$$\tau = \mu m g r = I\alpha = (m r^2).\, \alpha$$

or $$\alpha = \frac{\mu g}{r}.$$

$$\therefore \quad \omega = \omega_0 - \frac{\mu g}{r}\, t = \omega_0 - \frac{\mu g}{r} \cdot \frac{v}{\mu g} = \omega_0 - v/r.$$

When there is no slip, $v = \omega r$.

$$\therefore \quad v/r = \omega_0 - v/r \quad \text{or} \quad \frac{2v}{r} = \omega_0$$

or $$v = \tfrac{1}{2}\, \omega_0 r. \qquad \textit{Ans.}$$

5. (a) Since there is no slipping, the velocity of the point of contact of the string with the pulley is the vector sum of the velocity of the point of contact due to rotation plus the velocity of the centre of the mass of the spool. That is,

$$v_B = v_{cm} - \omega r.$$

But $v_{cm} = \omega R$. $$\therefore \quad v_B = v_{cm} - v_{cm}\, \frac{r}{R} = v_{cm}\, \frac{R - r}{R}.$$

This is also the velocity (v) of the falling mass.

$$\therefore \quad v = v_B = v_{cm} = \frac{R-r}{R}.$$

Considering conservation of energy when m lowers by h,

$$mgh = \tfrac{1}{2}mv^2 + (\tfrac{1}{2}Mv^2_{cm} + \tfrac{1}{2}I\omega^2)$$

$$= \frac{1}{2}mv^2 + \left(\frac{1}{2}Mv^2_{cm} + \frac{1}{2}\cdot\frac{1}{2}mr^2\frac{v^2{}_{cm}}{r^2}\right)$$

$$= \tfrac{1}{2}mv^2 + \tfrac{3}{4}Mv^2_{cm}$$

$$= \frac{1}{2}mv^2 + \frac{3}{4}M.\,v^2.\left(\frac{R}{R-r}\right)^2$$

or

$$v^2 = \frac{4mgh}{2m + 3M\left(\frac{R}{R-r}\right)^2}.$$

From $v^2 - v_0{}^2 = 2fs$, $\qquad v^2 = 2ah$

or

$$a = \frac{v^2}{2h} = \frac{2gh}{2m + 3M\left(\frac{R}{R-r}\right)^2}. \textbf{ Ans.}$$

(b) I (merry-go-round) $= 10 \times 1{\cdot}5^2 = 22{\cdot}5$ kgm².
I (of child) $= 30 \times 2 = 60$ kgm².

Angular momentum of the child about the axis of rotation before he jumps on $= (60 \times 5) \times 2$ (from the fact that moment of momentum is also angular momentum)

$$= 600.$$

By 'conservation of angular momentum'

$$600 = (22{\cdot}5 + 60)\omega$$

or

$$\omega = \frac{600}{82{\cdot}5} = 7{\cdot}27 \text{ rad/s.} \quad \textbf{Ans.}$$

6. (a) Since the centre of mass of the spool remains stationary, there is no change in its potential energy as the mass 'm' falls.

Let v be the velocity of the mass 'm' when it has gone down by h. This is also the velocity of the point of contact of the string with the connecting tube. Therefore, angular velocity $= v/r$.

From conservation of energy

$$mgh = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2$$

(there is no linear kinetic energy of the spool)

$$= \frac{1}{2}mv^2 + \frac{1}{2}\cdot\left(\frac{1}{2}MR^2\right)\cdot\frac{v^2}{r^2}$$

$$= \frac{1}{2}mv^2 + \frac{1}{4}Mv^2\frac{R^2}{r^2}$$

or $$4mgh=2mv^2+Mv^2\,R^2/r^2$$

or $$v^2=\frac{4mgh}{2m+M\frac{R^2}{r^2}}=\frac{2gh}{1+\frac{M}{m}\cdot\frac{R^2}{2r^2}}$$

$$=2ah$$ where a = acceleration of fall of the mass.

$$a=\frac{g}{1+\frac{M}{m}\;\frac{R^2}{2r^2}}.$$

From the motion of 'm', $mg-T=ma$

or $$T=mg-ma=mg\left(1-\frac{1}{1+\frac{M}{m}\;\frac{R^2}{2r^2}}\right)=mg\;\frac{\frac{M}{m}\cdot\frac{R^2}{2r^2}}{1+\frac{M}{m}\;\frac{R^2}{2r^2}}$$

For the stationary position of the spool we have

$$T-Mg\sin\alpha=0$$

or $$mg\;\frac{\frac{M}{m}\;\frac{R^2}{r^2}}{1+\frac{M}{m}\;\frac{R^2}{2r^2}}=Mg\sin\alpha$$

or $$\sin\alpha=\frac{1}{\frac{r^2}{R^2}+\frac{M}{2m}}.$$ *Ans.*

(b) The velocity of the centre of mass of the tractor wheel is v

∴ Velocity of the lower half of the belt at any instant

$$=v-v=0.$$

Velocity of the upper half $=v+v=2v$.

∴ $$T=\frac{1}{2}\cdot\left(\frac{m}{2}\right)\cdot(2v)^2=mv^2.$$ *Ans.*

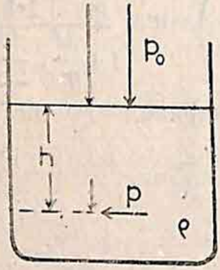
* * *

LESSON 8

Topics : **Hydrostatics and Hydrodynamics**

8.1 Hydrostatics

Pressure at a point inside a liquid :



$$p=p_0+\rho gh$$

where p_0 is the atmospheric pressure, ρ is the density of the liquid and h is the depth of the point below the free surface.

Pressure is a scalar.

The unit of pressure may be 'atmosphere' or 'cm of mercury'. These are derived units. The absolute unit of pressure is N/m^2. Normal atmospheric pressure, i.e., 76 cm of mercury, is approximately equal to 10^5 N/m^2.

Thrust. Thrust = pressure × area.

Thrust has the unit of force.

Laws of Liquid Pressure. (a) A liquid at rest exerts pressure equally in all directions.

(b) Pressure at two points on the same horizontal line in a liquid at rest is the same.

(c) Pressure exerted at a point in a confined liquid at rest is transmitted equally in all directions and acts normally on the wall of the containing vessel. This is called Pascal's law.

A hydraulic press works on this principle of transmission of pressure.

Principle of floating bodies (Law of Floatation) $W=W'$, that is, weight of body = weight of displaced liquid or buoyant force.

Weight of displaced liquid is also called buoyancy or upthrust.

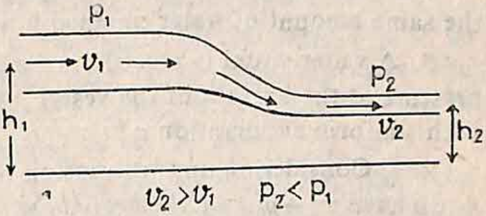
Hydrometers work on the principle of floating bodies. This principle may also be applied to gases (e.g. balloon).

Liquids and gases are together called fluids. The important difference between them is that liquids cannot be compressed while gases can be compressed. Hence, the density of a liquid is the same everywhere and does not depend on its pressure. For a gas, the density is proportional to its pressure.

8.2 Hydrodynamics

Bernoulli's theorem : $\frac{1}{2}v^2+gh+\frac{p}{\rho}=$ a constant for a streamline flow of a fluid (liquid or gas).

Here, v is the velocity of the fluid, h is its height above some horizontal level, p is the pressure and ρ is the density.



According to this principle, greater the velocity the less is the pressure in a fluid flow.

It is useful to remember that in liquid flow, the volume of liquid flowing past any point per second is the same for every point. Therefore, when the cross-section of the tube decreases, the velocity increases.

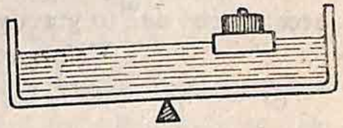
Note. Density = relative density or specific gravity × 1000 kg/m^3.

CRITICAL QUESTIONS

1. A wooden cylinder floats in water in a vessel with its axis vertical. How will the level of water in the vessel change if the cylinder floats with its axis horizontal.

Ans. No change in the level of water because in both the positions the cylinder displaces the same volume of water.

2. A vessel filled with water is supported on a knife edge. Will equilibrium be violated if a small board carrying a weight is placed on the surface of the water ?



Ans. No. Equilibrium will not be violated, since according to Pascal's law the pressure on the bottom is the same everywhere.

3. A match box with a small piece of stone inside it floats in a cup of water. Will the level of the water in the cup change if the stone is taken out of the box and dropped into the water ?

Ans. The level of the water in the cup will go down. See mathematical proof in Introductory Physics, Part I, Chapter 13, General Physics.

4. A boat floats in water of a swimming pool. Water from a well is pumped into the pool. When will the pumps perform more work : when a big boat floats in the pool, or a small toy-boat ?

Ans. In both the cases the pumps perform the same work, since the same amount of water pumped in rises to the same level.

5. A water vessel is placed on the floor of an elevator. Will the pressure at the bottom of the vessel change if the elevator goes up with uniform acceleration a ?

Ans. Considering upward motion of a column of liquid of depth h, we have $pA-p_0A-(Ah\rho)g=(Ah\rho)a$

or
$$p=p_0+h\rho(g+a).$$

When the elevator is at rest
$$p_i=p_0+h\rho g$$
$$\therefore \quad \triangle p=p-p_i=h\rho a.$$

Thus the pressure at the bottom increases.

6. A wooden cylinder floats in water in a vessel placed on the floor of an elevator. The length of the cylinder outside water is l. If the elevator goes down with uniform acceleration a, will l change ?

Ans. When there is no motion

$AL\rho g=(L-l)A\rho_w g$, where L = total length of cylinder
A = area of cross-section
ρ = density of cylinder
ρ_w = density of water

or $L\rho=(L-l)\rho_w$.

When the elevator goes down with accelaration a, the effective acceleration due to gravity on both cylinder and water is $(g-a)$.

$$\therefore \quad AL\rho(g-a)=(L-l')A\rho_w(g-a)$$

or
$$L\rho=(L-l')\rho_w.$$
$$\therefore \quad l=l'.$$

So the cylinder will neither sink nor rise.

7. Mercury is poured into a vertical U-tube, and water is poured in above it. The level of water is the same in both arms. Will the

level of the water and the mercury be the same if a piece of wood is dropped into one arm and some water equal in weight to this piece is added to the other ?

Ans. Dropping of the piece of wood is exactly the same as adding of the amount of water displaced by it, which in its turn is equal to the weight of the body. Hence, if cross-sections of the arms are the same, the levels of water in both arms will stand at the same horizontal level.

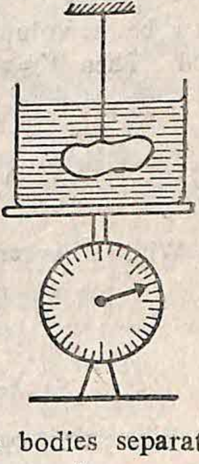
8. An air mattress is filled with air to a pressure greater than the atmospheric pressure. When will the air pressure in the mattress be greater : when a man stands on it or when he lies on it ?

Ans. If the man stands on the mattress, his weight will be distributed over a smaller area, that is, the area of his feet than when he lies down. Hence, the air pressure in the mattress will be greater when the man stands than when he lied on it.

9. A man carries a tyre tube and decides to make it lighter by making use of buoyancy of air. For this purpose he inflates the tube, thus increasing its volume. Will his aim be achieved ?

Ans. No, his aim will not be achieved. Increased buoyancy of air will be made up by weight of air blown in. Since density of compressed air is generally greater than the density of atmospheric air, he will disserve his aim rather than serving his aim.

10. A tank containing water is placed on a spring balance. A stone of weight *w* is hung and lowered into the water without touching the sides and the bottom of the tank. Explain how will the reading change.



Ans. We make free-body diagrams on the bodies separately and consider their equilibrium. Like all other forces buoyancy is also exerted equally on the two bodies in contact. Hence water exerts buoyant force, say, *B* on the stone upward and the stone exerts on the water the same force downward. The forces acting on the 'water+container' system are : W, weight of the system downward; B, buoyant force of the stone downward; and the force 'R' of the spring in the upward direction. For equilibrium we have

$$R=W+B.$$

Thus the reading of the spring scale will increase by an amount equal to the weight of the liquid displaced, that is, by an amount equal to the buoyant force.

Note. Weight of the stone will not come in the free-body diagram of the 'water+container' system because the weight is the action and reaction between the stone and the earth and not between the stone and the water.

11. A ball floats on the surface of water in a container exposed to the atmosphere. Will the ball remain immersed at its former depth or will it sink or rise somewhat if (a) the container is covered and the air is removed, (b) the container is covered and the air is compressed ?

Ans. Let v be the volume of the ball above the level of water and V is its total volume. W=weight of the ball. Then remembering that, strictly speaking, air also exerts buoyant force, we have

$$W=v\rho_0 g+(V-v)\rho_w g$$

where ρ_0=density of atmospheric air, ρ_w=density of water.

Let v' be the volume above water level when density of air is changed. Then $W=v'\rho g+(V-v')\rho_w g$ where ρ=density of air.

$$\therefore \quad v\rho_0 g+(V-v)\rho_w g=v'\rho g+(V-v')\rho_w g$$

or

$$v'=v\frac{\rho_w-\rho_0}{\rho_w-\rho} \quad \text{or} \quad \frac{v'}{v}=\frac{\rho_w-\rho_0}{\rho_w-\rho}.$$

(a) When air is removed, $\rho=0$

$$\frac{v'}{v}=\frac{\rho_w-\rho_0}{\rho_w},$$

$\therefore \quad v'<v.$ *So the ball sinks.*

(b) When air is compressed, $\rho>\rho_0$ or $\rho_w-\rho<\rho_w-\rho_0$

or

$$1<\frac{\rho_w-\rho_0}{\rho_w-\rho}$$

$\therefore \quad \frac{v'}{v}>1$ or $v'>v$. So the ball rises.

12. A solid cylinder is placed erect in a container in contact with the base. When liquid is poured into the container, none of it goes beneath the solid, which remains closely in contact with the base. Is there a buoyant force on the solid ? Explain.

Ans. No. There is no buoyant force on the solid because the liquid is not in contact at the bottom and so it exerts no upward thrust on it.

13. An open bucket of water is on a smooth inclined plane forming an angle α to the horizontal. How will the level rest relative to the plane when the bucket is allowed to slide down the plane ?

Ans. The level will remain parallel to plane during the journey down the plane. The component of the force mg down the plane, which is $mg \sin\alpha$ produces acceleration down the plane. The surface is subjected to force $mg \cos\alpha$ perpendicular to the plane. A liquid surface keeps itself at right angles to the force to which it is subjected. So the level rests parallel to the plane during the downward journey of the bucket.

14. If a U-tube containing water is rotated about a vertical axis through the centre of one limb, the level will fall in one limb and rise in the other. Explain carefully.

Ans. In a rotating liquid there is essentially a 'horizontal gradation of pressure' because without such gradation rotational equilibrium of the liquid is not possible. The pressure increases with distance from the axis. For deduction see Lesson 5. This is why there is fall of level at the axis and rise of the level in the other arm.

15. Explain why a uniform wooden stick which will float horizontally if it is not loaded will float vertically if enough weight is added at one end.

Ans. A floating body floats in stable equilibrium when its centre of gravity is well below the centre of buoyancy, that is, the centre of gravity of the displaced liquid. When loaded by enough weight the centre of gravity goes well below the centre of buoyancy.

16. A piece of ice floats in a glass filled with water. How will the level of the water in he glass change as the ice melts ? Consider the following cases :

(a) ice is homogeneous,

(b) a stone is frozen in the ice,

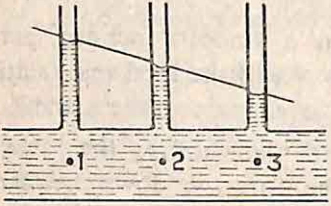
(c) the ice contains an air bubble.

Ans. (a) The level will not change.

(b) The level of water in the glass will drop.

(c) The level of water will not change.

See mathematical proof in Introductory Physics, Part I, Chapter 13, General Physics.



17. The height of the liquid in the standpipes indicates that the pressure drops along the channel, even though the channel has a uniform cross-section and the liquid flowing from left to right is incompressible. Explain.

Ans. Bernoulli's theorem is applicable to incompressible, non-viscous fluid free from external forces. But in actual liquid flow, there is some external and internal friction, i.e., viscous force. When these are taken into account, Bernoulli's theorem is modified to $\frac{1}{2}v^2+\frac{p}{\rho}+gh+W=$ a constant where W is the work done per unit mass by the fluid against frictional forces. Here v and h are the same at all points.

$$\therefore \quad \frac{p_1}{\rho}+W_1=\frac{p_2}{\rho}+W_2 \quad \text{or} \quad W_2-W_1=\frac{p_1-p_2}{\rho}$$

$$\because \quad W_2>W_1, \quad \therefore \quad p_1>p_2.$$

18. It is found that liquid will flow faster and more smoothly from a sealed can when two holes are punctured in the can than when one hole is made. Explain.

Ans. When two holes are made, the velocity of flow through each is much less than the velocity of flow when only one hole is there. Hence flow becomes streamline with two holes. This is the reason why liquid flows more smoothly and faster with two holes.

19. Two row boats moving parallel to one another in the same direction are pulled towards one another. Explain.

Ans. When they are close to each other, the velocity of water in between them increases, resulting in fall of pressure there according to Bernoulli's theorem. The pressures from the sides push them together.

20. Can the action of a parachute in retarding free fall be explained by Bernoulli's theorem ?

Ans. When the parachute opens up, the pressure of air above drops and so upward thrust is called into play to balance the weight of the parachutist.

21. Can you explain why an object falling from a great height reaches a steady terminal velocity ?

Ans. The viscous force on a body depends on its velocity.

The greater the velocity the greater is the viscous force. When a body falls from sufficient height, it acquires sufficient velocity to produce enough viscous force to balance its weight. The resultant force on the body being zero, the body moves with uniform velocity, called terminal velocity.

22. On take off would it be better for an airplane to move into the wind or with the wind ? On landing ?

Ans. In 'take off' of an airplane it needs maximum dynamic lift on it, which is associated with an unsymmetrical set of streamlines relatively close together on the upper side and relatively far apart below. This unsymmetric distribution of streamlines can better be obtained by moving into the wind. On 'landing' it should move with the wind when streamlines will be uniformly distributed on the two sides of the airplane and so no lift will arise and the plane will land under gravity.

23. Does the difference in pressure between the lower and upper surfaces of an airplane wing depend on altitude of the moving plane ? Explain.

Ans. Yes, we have $\frac{1}{2}v_1^2+\frac{p_1}{\rho}=\frac{1}{2}v_2^2+\frac{p_2}{\rho}$

$$\triangle p=p_2-p_1=\tfrac{1}{2}\rho(v_1^2-v_2^2).$$

At high altitudes density is considerably low and so the pressure difference will depend on the altitude of the airplane.

***24**. If p is the pressure of gas inside the exhaust chamber of a rocket and p_0 is the pressure of the gas outside the chamber, the forward thrust on the rocket is $2a(p-p_0)$ instead of $a(p-p_0)$, where a is the area of the orifice. Why does the factor "2" appear ?

Ans. The formula 'thrust = pressure difference × area' holds for fluids at rest. This does not hold for fluid in motion. By Bernoulli's theorem (a theorem applicable to fluid in streamline motion) $\frac{p}{\rho}+\frac{1}{2}V^2=\frac{p_0}{\rho}+\frac{1}{2}v^2$ and $AV=av$ where A = area of the chamber, a = area of the orifice. Since $A \gg a$, $v \gg V$, hence V^2 may be neglected in comparison to v^2.

$$\therefore \quad v^2 = 2\frac{(p-p_0)}{\rho}.$$

Now thrust on the rocket $= v\dfrac{dM}{dt}$.

But the mass (dM) of gas flowing out in time dt is ($av\rho dt$).

$$\therefore \quad \frac{dM}{dt} = av\rho.$$

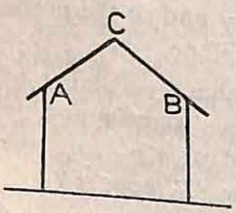
$$\therefore \quad \text{Thrust} = av^2\rho = a\rho \times \frac{2(p-p_0)}{\rho} = 2a(p-p_0).$$

25. The destructive effect of a tornado (twister) is greater near the centre of the disturbance than near the edge. Explain.

Ans. Near the edge the velocity is zero and the pressure is the normal atmospheric pressure. At the centre velocity is maximum and so there is maximum drop of pressure at the centre. Thus pressure difference between the central region with the surrounding being maximum, the destructive power of tornado is maximum at the centre.

26. Why will a light ping-pong ball dance over a stream of a gas or water issuing at high speed from a tube ?

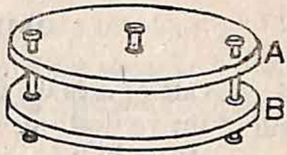
Ans. Since the stream has a high velocity, the pressure inside the stream is below the atmospheric pressure. The ball is supported from the bottom by the thrust of the stream and kept in position by the lateral pressure difference.



27. During storms the strong winds tear off the roofs of thatched houses along the ridge C if the roof is fastened more firmly at the edges A and B (see fig.) than at the ridge. On the other hand the roof is secured more firmly at the ridge C than at the edges, the wind will first lift the roof up and then carry it aside. Explain.

Ans. The pressure of the air streaming over the roof is less than that of the stationary air below the roof. This is according to Bernoulli's theorem. The surplus pressure of the stationary air of the room acts normally on the two inclined halves of the roof.

When the roof is secured at the edges *A* and *B*, the thrusts on the two halves will have turning effects about the edges and so the roof is likely to tear along the ridge. On the other hand if the roof is secured rigidly along the ridge, the thrusts then will lift the roof against gravity only to be carried away by the wind.



28. A device (see figure) consists of two discs *A* and *B*. The lower disc *B* hangs on three pins fixed to *A* along which it can freely move up and down. The upper disc *A* is provided with a pipe at its centre. If a compressed air stream is passed through the pipe, the lower disc begins to knock the upper one. Explain, why ?

Ans. When air is blown through the pipe, the air between the discs are set in motion, resulting in heavy drop of pressure. The surplus pressure from below pushes the disc *B* and knocks it against the upper disc.

29. Why do tea leaves gather in the middle of a glass after stirring ?

Ans. Tea leaves are suspended particles having density less than density of water. Due to rotation of the water in glass, there is a 'gradation of pressure along' the horizontal direction. The thrust on suspended particles due to pressure difference is inward for lighter particle. Hence the leaves gather in the middle of the glass vessel after stirring.

30. Liquids leak through a pin hole at the bottom of a vessel. Will kerosene oil leak faster than water ? Give reasons for your answer (assume same viscosity).

Ans. Both will leak at the same speed. The velocity of efflux does not depend on the density of liquid. The velocity is $\sqrt{2gh}$ where h is the height of the liquid. See deduction in Introductory Physics, Part I, Chapter 14, General Physics.

31. When a stopper is removed from a filled basin, the water drains out while circulating like a small whirlpool. The angular velocity of a fluid element about a vertical axis through the orifice appears to be greatest near the orifice. Explain.

Ans. In the curved flow of a fluid the product of velocity and radius remains constant. See deduction in Ex. 11 of this lesson.

That is, vr = a constant or ωr^2 = a constant.

Therefore, $\omega \propto \frac{1}{r^2}$. This is why angular velocity at the axis is maximum.

OBJECTIVE QUESTIONS

1. A plate of area 0·5 m² is immersed in water vertically. The length of the vertical side is ·5 m. The thrust on the plate due to water ($g = 10$ m/s²) is

(a) 1000 N/m², (b) 500 N/m², (c) 200 N/m², (d) 1250 N/m².

Ans. (d). Thrust = pressure at the centre × area

$$= \rho g(\tfrac{1}{2}h)\times A = \tfrac{1}{2}\rho g h A = \tfrac{1}{2}\times 1000\times 10\times \cdot 5\times \cdot 5 \text{ N/m}^2$$
$$= 1250 \text{ N/m}^2.$$

2. Two substances are mixed in equal volumes. Their densities are ρ_1 and ρ_2. The density of their mixture, assuming no contraction, is

(a) $\frac{\rho_1+\rho_2}{P_1P_2}$, (b) $\frac{\rho_1+\rho_2}{2}$, (c) $\sqrt{\rho_1\rho_2}$, (d) $\frac{\rho_1\rho_2}{\rho_1+\rho_2}$.

Ans. (b). $2V\rho = V\rho_1 + V\rho_2$ or $\rho = \frac{\rho_1+\rho_2}{2}$.

3. Two liquids are mixed in equal weights. Their densities are ρ_1 and ρ_2. The density of their mixture, assuming no contraction, is

(a) $\frac{\rho_1+\rho_2}{\rho_1\rho_2}$, (b) $\frac{\rho_1+\rho_2}{2}$, (c) $\sqrt{\rho_1\rho_2}$, (d) $\frac{\rho_1\rho_2}{\rho_1+\rho_2}$.

Ans. (d). $\frac{W}{\rho} = \frac{W}{\rho_1} + \frac{W}{\rho_2}$ or $\rho = \frac{\rho_1\rho_2}{\rho_1+\rho_2}$.

4. A beaker containing water is placed on the platform of a spring balance. The balance reads 1·5 kg. A stone of mass ½ kg and volume 0·001 m³ is immersed in water without touching the wall of the beaker. The balance reads

(a) 2 kg, (b) 1·5 kg, (c) 1·0 kg, (d) 2·5 kg.

Ans. (d). The scale reading = Weight of beaker + buoyant force

$$= 1\cdot 5 + 0\cdot 001 \times 1000 = 2\cdot 5 \text{ kg}.$$

Note. The weight of the stone does not affect the reading.

5. A vessel is filled with a liquid of density ρ up to a height h.

The velocity with which liquid emerges from a small hole at the bottom is

(a) $\sqrt{g\rho h}$, (b) $\sqrt{gh}$, (c) $\sqrt{2gh}$, (d) ρgh.

Ans. (c).

Note. Velocity of efflux does not depend on the density of liquid.

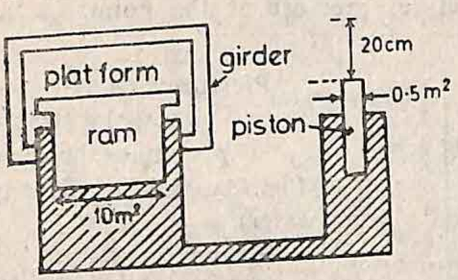
6. The gate of a canal is 5 m wide, and the water levels on either side of it are 20 m and 12 m. The resultant thrust on the gate ($g=10$ m/s^2) is

(a) $6A\times10^5$N, (b) 64×10^3N, (c) 64×10^4N, (d) 64×10^7N.

Ans. (a). Resultant thrust = difference of thrusts on the sides

$$=\tfrac{1}{2}pgh_1A_1-h_2\,pgh_2A_2$$
$$=\tfrac{1}{2}pg(20\times20\times5-12\times12\times5)$$
$$=\tfrac{1}{2}\times1000\times10\times1280=64\times10^5\text{N}.$$

7. The areas of the piston and ram of a hydraulic press are respectively 0·5 m^2 and 10 m^2. If a force of 10 kgf is applied on the piston, the force on the ram is



(a) 20 kgf, (b) 500 kgf, (c) 200 kgf, (d) 50 kgf.

Ans. (c). Pressure exerted $=\dfrac{10}{\cdot5}=20$ kgf/m^2.

Thrust produced = pressure × area $=20\times10=200$ kgf.

8. In the above example if the ram has space above it in between the girder and its platform to move up, the distance through which the ram moves when the piston is driven through 20 cm is

(a) 1 cm, (b) 5 cm, (c) ·5 cm, (d) ·1 cm.

Ans. (a). By assuming that liquid is incompressible

$$A_1l_1=A_2l_2,$$
$$0{\cdot}5\times{\cdot}2=10\times l_2 \quad\text{or}\quad l_2={\cdot}01\text{ m}=1\text{ cm}.$$

9. Mercury is poured into communicating vessels of different cross sections. An iron piece is dropped into the broad vessel.

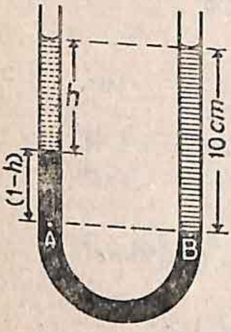
The level of mercury (a) will go down by unequal amounts, (b) will go down by equal amounts, (c) will go up by unequal amounts, (d) will go up by equal amounts.

Ans. (d). Since the pressure on the horizontal portion is the same at all points, the heights of mercury above the horizontal portion must also be the same.

EXAMPLES

1. A vertical U-tube of uniform cross-section contains mercury in both its arms. A glycerine (relative density 1·3) column of length 10 cm is introduced into one of the arms. Oil of density 800 kg/m^3 is poured in the other arm until the upper surface of the oil and glycerine are in the same horizontal level. Find the length of the oil column. Density of mercury $13{\cdot}6\times10^3$ kg/m^3.

Sol. Draw a horizontal line through the 'mercury-glycerine surface'. This is a horizontal line in the same liquid at rest, namely, mercury. Therefore, pressure at the points *A* and *B* must be the same.



Pressure at *A*

$$=p_0+{\cdot}1\times(1{\cdot}3\times1000)\times g$$

p_0=atmospheric pressure (density = relative density × density of water).

Pressure at *B*

$$=p_0+h\times800\times g+({\cdot}1-h)\times13{\cdot}6\times1000\times g$$

$$\therefore\quad p_0+{\cdot}1\times1300\times g=p_0+800gh+1360g-13600\times g\times h$$

or $\quad 130=800h+1360-13600h$

or $$h=\frac{1230}{12800}={\cdot}096\text{ m}=9{\cdot}6\text{ cm.}\qquad \textit{Ans.}$$

2. A cylinder of brass 10 cm high and 8 cm^2 in cross-sectional area is suspended in water by means of a string, so that its upper surface is horizontal and 5 cm below the surface of water. Find (a) the force on the top of the cylinder, (b) the force acting on the bottom of it, (c) the buoyant force on it.

Sol. (a) F_1 (force on top) = pressure × area $=(\rho gh)\times A$

$$=1000\times9{\cdot}8\times{\cdot}05\times8\times10^{-4}$$

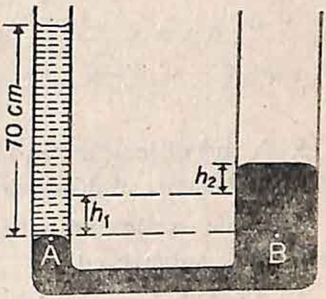
$$={\cdot}392\text{ N.}\qquad \textit{Ans.}$$

(b) F_2 (force on the bottom) $=1000\times 9{\cdot}8\times{\cdot}15\times 8\times 10^{-4}$
$={1{\cdot}176}$ N. *Ans.*

(c) Buoyant force = difference of (a) and (b) $={\cdot}784$ N. *Ans.*

3. Two communicating cylindrical tubes contain mercury. The diameter of one is two times than the other. A column of water of height 70 cm is poured into the narrow vessel. How much will the mercury level sink in this vessel and how much will it rise in the other vessel ? How much will the mercury level rise in the narrow vessel if a column of the same height is poured into the broad vessel ?

Sol. Let the level in the narrow tube go down by h_1 from the initial position (shown by the dotted line) in the narrow vessel and that in the wider vessel go up by h_2.



By the principle of incompressibility of liquid

$$\pi r^2 h_1=\pi 4r^2 h_2{}^2$$

or $$h_1=4h_2 \qquad \cdots \quad (1).$$

Draw a horizortal line through 'water-mercury surface' and consider two points A and B on it in the narrow and wide vessel respectively.

$$\therefore \quad \underset{\text{(pressure at } A)}{1000\times 9{\cdot}8\times{\cdot}7+p_0}=\underset{\text{(pressure at } B)}{13{\cdot}6\times 1000\times(h_1+h_2)\times 9{\cdot}8+p_0}$$

or $$700=13600(h_1+h_2)$$

or $$7=136(h_1+h_2) \qquad \cdots \quad (2).$$

Solving (1) and (2),

$$h_1={\cdot}04 \text{ m}=4 \text{ cm and } h_2={\cdot}01 \text{ m}=1 \text{ cm}. \quad \textit{Ans.}$$

When water is poured into the wider vessel, let the mercury level go down by h_2 in this tube and go up by h_1 in the narrow vessel. Again $\pi r^2 h_1=\pi 4r^2 h_2$ or $h_1=4h_2$.

Now B is at the 'water-mercury surface'.

$$\underset{\text{(pressure at } B)}{1000\times 9{\cdot}8\times{\cdot}7+p_0}=\underset{\text{(pressure at } A)}{13{\cdot}6\times 10000\times(h_1+h_2)\times 9{\cdot}8+p_0}$$

$$7=13(h_1+h_2).$$

Solving, $h_1={\cdot}04$ m $=4$ cm, $h_2={\cdot}01$ m $=1$ cm. *Ans.*

4. A rubber ball of mass m and radius 'r' is submerged into water to a depth h and released. What height will the ball jump up o above the surface of the water ? Neglect the resistance of water and air.

Sol. Let the ball go up by x above the level of water.

Let us now consider 'energy conservation' between the initial and final position. In both the positions kinetic energy of the body is zero. The potential energy in the first position with reference to the water level is $-mgh$ plus the work done by an external agent against the buoyant force which is $\left(\frac{4\pi}{3}r^3\rho g\right)h$, where ρ is the density of the water.

$$\therefore \quad -mgh+\left(\frac{4\pi}{3}r^3\rho g\right)h=mgx.$$

$$\therefore \quad x=\frac{\frac{4\pi}{3}r^3\rho-m}{m}\times h \quad \textit{Ans.}$$

5. A rod of length 6 m has a mass of 12 kg. It is hinged at one end at a distance of 3 m below the water surface.

(i) What weight must be attached to the other end so that 5 m of the rod are submerged ?

(ii) Find the magnitude and direction of the force exerted by the hinge on the rod. The specific gravity of the material of the rod ts 0·5.

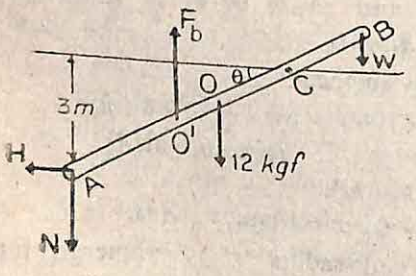
(I.I.T. 1976)

Sol. Mass per unit length of the rod is 2 kg. Therefore, mass of the submerged portion is 10 kg and its volume $=\frac{10}{500}$ m^3

(using the simple formula, volume $=\dfrac{\text{mass}}{\text{density}}$

and density = specific gravity × 1000 kg/m^3).

Therefore, buoyant force $(F_b)=\frac{10}{500}\times 1000=20$ kgf.



Let N and H be the vertical downward and horizontal reactions of the hinge on the rod. Considering 'horizontal and vertical translational equilibrium of the rod' we have $N+12+W=20$ (where W is the weight to be attached and $H=0$)

or
$$\left.\begin{aligned} N+W&=8\\ H&=0 \end{aligned}\right\} \quad \cdots \quad \text{(i)}$$

Considering 'rotational equilibrium' of the rod about A we have,

$$-20\times g\times 2{\cdot}5\cos\theta+12\times g\times 3\cos\theta+W\times 6\cos\theta=0$$

$$6W=50\text{ g}-36\text{ g}=14\text{ g}$$

$$W=\tfrac{7}{8}\text{ g}=\tfrac{7}{3}\text{ kgf.}\quad \textit{Ans.}$$

$$N=8-\tfrac{7}{3}=\tfrac{17}{3}\text{ kgf.}\quad \textit{Ans.}$$

6. A cylinder of area 300 cm and length 10 cm made of material of specific gravity 0·8 is floated in water with its axis vertical. It is then pushed downward, so as to be just immersed. Calculate the work done by the agent who pushes the cylinder into the water.

Sol. Weight of the cylinder

$$=(300\times 10^{-4})\times(10\times 10^{-2})\times 800\text{ kgf}=2{\cdot}4\text{ kgf.}$$

Let x be the length of the cylinder inside the water. Then by law of floatation

$$2{\cdot}4\,g=(300\times 10^{-4}\times x)\times 1000g$$

$$x={\cdot}08\text{ m.}$$

When completely immersed,

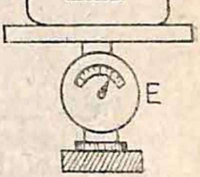
$$F_b\text{ (buoyant force)}=(300\times 10^{-4}\times{\cdot}1)\times 1000\times g=3g\text{ N.}$$

Thus to immerse the cylinder inside the water the external agent has to push it by ·02 m against average upward thrust.

Increase in upward thrust $=3g-2{\cdot}4g=0{\cdot}6g$ newton.

Since this increase in upthrust takes place gradually from 0 to $0{\cdot}6g$, we may take the average upthrust against which work is done as $0{\cdot}3g$ newton.

$\therefore$ Work done $=0{\cdot}3g\times{\cdot}02={\cdot}0588$ joule. *Ans.*



7. Block A in the figure hangs from a spring balance D and is submerged in a liquid C contained in beaker B. The weight of the beaker is 1 kg, the weight of the liquid is 1·5 kg. Balance D reads 2·5 kg and balance E reads 7 5 kg. The volume of the block A is 0·003 m³.

(a) What is the density of the liquid ?

(b) What will each balance read if the block A is pulled up out of the liquid ?

Sol. First let us calculate the buoyant force on the block A.

$$F_b=0{\cdot}003\times\rho g={\cdot}003\,\rho g \text{ where } \rho=\text{density of the liquid.}$$

Remembering that like all other forces buoyant force is also exerted mutually between the solid and the liquid we have from the free body diagram of the 'water + beaker' system.

$W_C + W_B + F_b = S_E$ where S_E is the upward reaction of the spring of the balance on the system and this is the reading of the scale E.

$$\therefore \quad 1g + 1{\cdot}5g + {\cdot}003\rho g = 7{\cdot}5g$$

or $\quad {\cdot}003\rho = 5{\cdot}0 \qquad$ or $\quad \rho = 1666{\cdot}7$ kg/m³. *Ans.*

(b) When A is taken out, D will read the weight of A and E will read the weight of 'beaker + water'.

$\therefore$ Reading of $E = 1$ kg $+ 1{\cdot}5$ kg $= 2{\cdot}5$ kg. *Ans.*

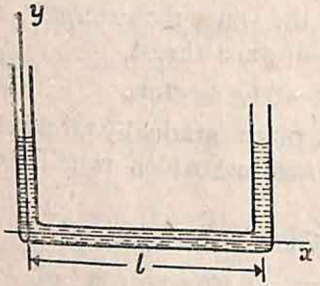
Let W_A be the weight of A, considering its equilibrium of A in the liquid.

$$W_A - F_b = S_D$$

where S_D is the reading of D.

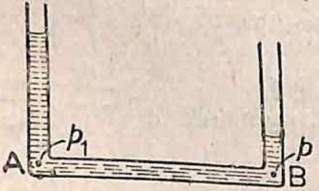
$$\therefore \quad W_A = F_b + S_D = \left({\cdot}003 \times \frac{5}{{\cdot}003}\, g\right) + 2{\cdot}5g = 7{\cdot}5g = 7{\cdot}5 \text{ kg}.$$

$\therefore$ Reading of balance D = Weight of $A = 7{\cdot}5$ kg. *Ans.*



8. A glass tube is bent into a rectangular U-tube and partially filled with a liquid. The length of the horizontal portion of the U is l. Find an expression for the difference in height h between tops of the liquid columns in the vertical arms (a) when the tube has a linear acceleration a in the direction of the x-axis, (b) when the tube has a uniform angular velocity about the y-axis.

Sol. (a) Consider the motion of the horizontal portion AB of the U-tube. Let p_1 and p_2 be the pressures of the liquid at A and B.



Then forward thrust on the liquid mass $= p_1A - p_2A$ where A is the area of cross-section of the tube.

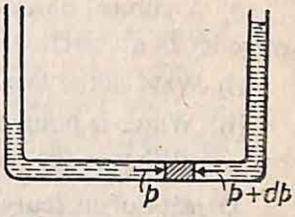
Mass of the liquid in $AB = lA\rho$.

$\therefore \quad p_1A - p_2A = (lA\rho)^a \qquad$ or $\quad p_1 - p_2 = l\rho a$.

Let h be the difference in liquid levels.

Then $p_1-p_2=h\rho g \quad \therefore \quad h\rho g=l\rho a$

or $$h=\frac{la}{g}. \quad \textit{Ans.}$$



(b) Let us consider an element dx at a distance x from the y-axis. Let p and $p+dp$ be the pressures on the sides of the element. Then the centripetal force on the element

$$=Adp.$$

Mass of the element $=(Adx\rho)$.

$$\therefore \quad Adp=(Adx\rho)\omega^2 x$$

(Formula used, $F=m\omega^2 r$)

or $$dp=\rho\omega^2 x\,dx$$

or $$p=\rho\omega^2.\tfrac{1}{2}x^2+c \text{ (a constant).}$$

At $x=0$, $p=p_1$ and at $x=l$, $p=p_2$.

$\therefore \quad p_1=c$ and $p_2=\frac{1}{2}\rho\omega^2 l^2+p_1$

or $$p_2-p_1=\tfrac{1}{2}\rho\omega^2 l^2.$$

But $$p_2-p_1=h\rho g \quad \therefore \quad h\rho g=\tfrac{1}{2}\rho\omega^2 l^2$$

or $$h=\frac{\omega^2 l^2}{2g}. \quad \textit{Ans.}$$

10. Air streams horizontally past an airplane wing of area 120 m^2 weighing 300 kg. The speed over the top surface is 60 m/s and 45 m/s under the bottom surface. What is the lift on the wing? The net force on it? Density of air $=1{\cdot}293$ kg/m^3.

Sol. By Bernoulli's theorem we have

$$\frac{1}{2}v_1^2+\frac{p_1}{\rho}=\frac{1}{2}v_2^2+\frac{p_2}{\rho} \quad \text{or} \quad p_2-p_1=\tfrac{1}{2}\rho(v_1^2-v_2^2)$$

$\therefore \quad \Delta p$ (difference in pressure) $=\frac{1}{2}\times 1{\cdot}293(60^2-45^2)=1018$ N/m^2.

$\therefore \quad$ The net upthrust $=1018\times 120=1{\cdot}22\times 10^5$ N

$=1{\cdot}25\times 10^4$ kgf. *Ans.*

$\therefore \quad$ The lift on the wing $=(1{\cdot}25\times 10^4-{\cdot}300)$ kgf

$=1{\cdot}22\times 10^4$ kgf. *Ans.*

EXERCISES

1. A piece of copper having an internal cavity weighs 264 gm in air and 221 gm in water. Find the volume of the cavity. Density of copper $=8{\cdot}8$ gm/cm^2. (I.I.T. 1963)

2. A stone of density 2·5 gm/c.c. completely immersed in sea water is allowed to sink from rest. Calculate the depth to which the stone would sink in 2 seconds. Neglect the effect of friction. Specific gravity of sea water is 1·025 and acceleration due to gravity is 980 cm/sec^2. (I.I.T. 1969)

3. A cubical block of iron 5 cm each on each side is floating on mercury in a vessel.

(i) What is the height of the block above the mercury level ?

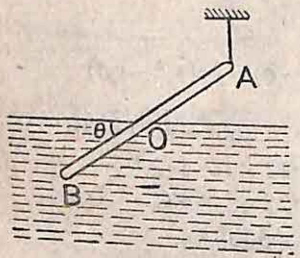
(ii) Water is poured into the vessel, until it just covers the iron block. What is the height of the water column ?

Density of mercury = 13·6 gm/cc, density of iron = 7·2 gm/cc. (N.D.A. 1981; I.I.T. '73)

4. (a) A body weighs m_1 in a fluid of density d_1 and m_2 in a fluid of density d_2. What would be its weight in a fluid of density d_3 ? [J.E.E. (W.B.) 1980]

(b) Calculate the percentage error arising from neglecting the buoyancy of air in weighing an object of relative density 12 with brass weights of density (relative) 8 by a physical balance. (Density of air = 1·2 kg/m^3). [J.E.E. (W.B.) 1981]

5. A common hydrometer, with a stem of 10 cm long and volume 1/10 of that of the whole hydrometer, floats with 2 cm length of its stem above the surface of water. What is the density of the liquid in which this hydrometer floats with 4 cm of its stem above the surface of the liquid ? (N.D.A. 1979)



6. A uniform rod AB, 4 m long and weighing 12 kg, is supported at end A with a 6 kg lead weight at B. The rod floats as shown in the figure with one half its length submerged. The buoyant force on the lead mass is negligible as it is of negligible volume. Find the tension in the cord and the total volume of the rod.

7. A hollow sphere of inner radius 9 cm and outer radius 10 cm floats half submerged in a liquid of specific gravity 0·8. Calculate the density of the material of which the sphere is made. What would be the density of a liquid in which the hollow sphere would just float completely submerged ?

8. A flat bottomed thin-walled glass tube has a diameter of 4 cm and it weighs 30 gm. The centre of gravity of the empty tube is 10 cm above the bottom. Find the amount of water which must be poured in the tube so that when it is floating vertically in a tank of water, the centre of gravity of the tube and its contents is at the mid-point of the immersed length of the tube. (I.I.T. 1968)

9. Mercury is poured into a cylindrical communicating vessels of different cross-sections. An iron piece of volume V_0 is dropped into the broad vessel, and as a result the level of mercury rises in it. Calculate the rise of the level in both the vessels if A is the area of cross-section of the smaller vessel, ρ_i = density of iron, ρ_m = density of mercury, ρ_w = density of water and A' = area of the wider vessel.

10. In the above example if water is poured into the wider vessel until the mercury reaches the previous level, what is the height h of the water column? Assume that water covers the iron piece completely.

11. The upper surface of water in a standpipe is a height H above evel ground. At what depth h should a small hole be put to make the emerging horizontal water stream strike the ground at the maximum distance from the base of the standpipe? What is the this maximum distance?

12. The speed of flow past the lower surface of a wing of an air plane in 50 m/s. What speed of flow over the upper surface will give a lift of 100 kgf/m^2? Take density of air = 1·20 kg/m^3.

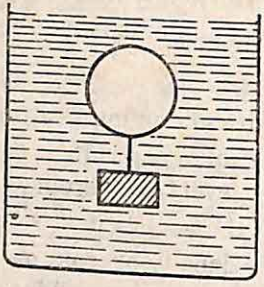
13. Consider the stagnant air at the front edge of a wing and the air rushing over the wing surface at a speed v. Find the greatest possible value for v in streamline flow, assuming that air is incompressible. Take the density of air 1·2 kg/m^3.

14. A liquid flows out of a broad vessel through a narrow vertical pipe How are the pressure and the velocity of the liquid in the pipe distributed when the height of the liquid level in the vessel is H from the lower end of the pipe and the length of the pipe is h.

15. A pump is designed as horizontal cylinder with a piston having an area A and an outlet orifice having an area of 'a' arranged near the cylinder axis. Find the velocity of outflow of the liquid from the pump if the piston moves with a constant velocity under the action of a constant force F. The density of the liquid is ρ.

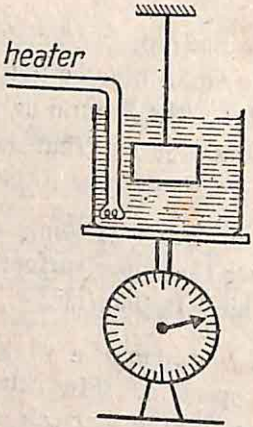
TEST YOURSELF 8

1. Answer the following questions— $15 \times 3 = 45$



(i) A balloon filled with air is weighed so that it barely floats in water as shown in the figure. Explain why it sinks to the bottom when it is submerged more by a short distance.

(ii) A block of wood is floating on water at 0°C with a certain volume V above water level. The temperature of water is slowly raised from 0°C to 20°C. How will the volume V change with the rise in temperature ? (I.I.T. 1974)



(iii) A body immersed in a liquid is balanced on scale. Will the reading on the scale be altered if the liquid be heated together with the body ?

(iv) A glass bulb is balanced by a brass weight in a sensitive beam balance. State what will happen when the balance is covered by a bell jar which is then evacuated. Explain. (I.I.T. 1972)

(v) A block of ice is floating in a liquid of specific gravity 1·2 contained in a beaker. When the ice melts completely, will the level in the beaker change ?

(vi) It is the practice of masons, while laying out and levelling up the foundations of buildings, to use long transparent plastic pipes. The theory is that the water, seeking a common level, will be of the same height in both the tubes and thus affect a level. What happens if a bubble of air is left in the pipe ?

(vii) When a steam boiler in which pressure of steam is 10 to 15 atm blows up, considerable damage is caused. Yet a hydraulic press, in which the pressure is much higher, blows up, not very great damage results from the explosion. Why ?

(viii) A hollow sphere and a solid sphere of the same radius and the same material fall through air from the same height. Which one arrive at the ground first ? (Neglect viscous effect of the air.)

(ix) Do rain drops of different sizes reach the earth with the same speed ? Explain.

(x) A spherical bob made of cork floats half submerged in a pot of tea at rest on the earth. Will the cork float or sink aboard a spaceship coasting in free space ? On the surface of Jupiter ?

(xi) Why is it easier to throw a curve with a tennis ball than with a base ball ?

(xii) Explain why is it dangerous to stand near a speedy train ?

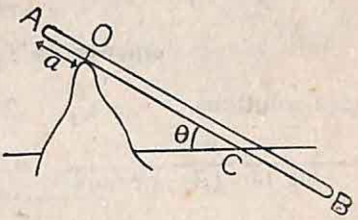
(xiii) A barge filled with scrap iron in a canal lock. If the iron is thrown over board, what happens to the water level in the lock ?

(xiv) Very often a sinking ship will turn over as it becomes immersed in water. Why ?

(xv) What is the pressure at a depth h below the free surface of a liquid falling freely ? 55

Answer the following—

2. One end of a board of length l is placed on the top of a stone projecting outside water. A part of length 'a' is above the point of support. What part of the board is below the surface of the water if the specific gravity of wood is s ? 10



3. A conical vessel of semi-vertical angle α but without a bottom stands tightly on a table. A liquid is poured into the vessel and as soon as its level reaches the height h, the pressure of the liquid raises the vessel. The radius of the bottom of the vessel is R and its weight is W. What is the density of the liquid ? 15

4. A piston weighing $W = 3$ kg has the form of a circular disc of radius $R = 4$ cm. The disc has a hole into which a thin-walled pipe of radius $r = 1$ cm is inserted. Initially the piston is at the bottom

of the cylinder. What height will the piston rise if $m = 700$ gm of water is poured into the pipe ? 10

5. A gas flows with a velocity v along a pipe of sectional area a 90° bend at A. What force does the gas exert on the pipe at A if its density is ρ ? Neglect compression of the gas and friction. 10

6. An expansible balloon filled with air floats on the surface of a lake with 2/3 of its volume submerged. How deep must it be sunk in the water so that it is just in equilibrium, neither sinking further nor rising. The height of water barometer $= 10$ m. 10

Full Marks 100

Your score.......

Answers of Exercises

1. 13 c.c. **2.** 11·564 m. **3.** $h = 2{\cdot}35$ cm; 2·54 cm.

4. $\dfrac{m_2(2d_2 - d_1 - d_3) - m_1(d_3 - d_2)}{d_2 - d_1}$; ·005%. **5.** 1·02. **6.** 32×10^3 c.c.; 19·6 N. **7.** 400 kg/m³. **8.** 110·6 gm. **9.** $x = \dfrac{V_0 \rho_i}{(A + A')\rho_m}$.

10. $h = \dfrac{\rho_m(\rho_i - \rho_w)V_0}{\rho_w(\rho_m - \rho_w)A}$. **11.** $H/2$. **12.** 64·3 m/s. **13.** $4{\cdot}1 \times 10^2$ m/s.

14. $v = \sqrt{2gH}$; $p_0 - \rho g(h - x)$. **15.** $\sqrt{\dfrac{2F}{A\rho}}$.

Answers to Test Questions

1. see solutions. **2.** $x = l - a - \sqrt{(l-a)^2 - ls(l - 2a)}$.

3. $\rho = \dfrac{W}{\pi h^2 g \tan\alpha(R - \frac{1}{3}h \tan\alpha)}$. **4.** 10 cm. **5.** $\sqrt{2av^2\rho}$. **6.** $H = 5$ m.

Solution to Exercises

1. Let v be the volume of cavity.

The external volume of the piece $=$ volume of water displaced

$$= \frac{{\cdot}264 - {\cdot}221}{1000}\,\text{m}^3.$$

The volume of matter (copper) $= \dfrac{{\cdot}264}{1000 \times 8{\cdot}8}$ m³.

Now, external volume $=$ volume of matter $+$ volume of cavity.

$$\therefore \text{ Volume of cavity} = \frac{\cdot 264 - \cdot 221}{1000} - \frac{\cdot 264}{8800}$$
$$= 13 \times 10^{-6} \text{ m}^3 = 13 \text{ c.c.} \quad \textit{Ans.}$$

2. Let v be the volume of the stone.

Then mass of the stone $= v \times (2{\cdot}5 \times 1000) = 2500v$ kg.

Effective downward force = weight − buoyant force

$$= 2500vg - (1{\cdot}025 \times 1000)vg$$
$$= (2500v) \times a$$

or
$$a = g\left(1 - \frac{1{\cdot}025 \times 10}{25}\right) = 5{\cdot}782 \text{ m/s}^2.$$

$$\therefore \quad h = \tfrac{1}{2} \times 5{\cdot}782 \times 2^2 = 11{\cdot}564 \text{ m.} \quad \textit{Ans.}$$

3. (i) Let h cm be the height above the mercury level and A = area of cross-section of the block.

Then by law of floatation

$$(5-h) \times 10^{-2} A \times 13{\cdot}6 \times 10^3\, g = 5 \times 10^{-2} A \times 7{\cdot}2 \times 10^3 \times g$$

(Formula used : weight = volume × density × g)

or
$$(5-h) \times 13{\cdot}6 = 7{\cdot}2 \times 5$$
$$h = 2{\cdot}35 \text{ cm.} \quad \textit{Ans.}$$

(ii) Let h cm be in water.

Then
$$(5-h) \times 10^{-2} \times A \times 13{\cdot}6 \times 10^3 \times g + h \times 10^{-2} \times A \times 1000\, g$$
$$= 5 \times 10^{-2} \times A \times 7{\cdot}2 \times 10^3\, g$$

$$(5-h)13{\cdot}6 + h = 5 \times 7{\cdot}2$$
$$5 \times 13{\cdot}6 - 5 \times 7{\cdot}2 = 12\ 6h$$

or
$$h = 2{\cdot}54 \text{ cm.} \quad \textit{Ans.}$$

4. (a) Let m be the mass and ρ be the density of the solid.

Then
$$m_1 g = mg - \frac{m}{\rho} d_1 g \quad \quad \text{.. (i)}$$

$$m_2 g = mg - \frac{m}{\rho} d_2 g \quad \quad \text{.. (ii)}$$

and
$$m_3 g = mg - \frac{m}{\rho} d_3 g \quad \quad \text{.. (iii).}$$

From (i) and (ii) by subtracting we have

$$\rho = \frac{d_2 - d_1}{m_1 - m_2} m.$$

Similarly from (ii) and (iii)

$$\rho = \frac{d_3 - d_2}{m_2 - m_3} . m$$

$$\therefore \quad \frac{d_2-d_1}{m_1-m_2}=\frac{d_3-d_2}{m_2-m_3}$$

or
$$m_3=m_2-\frac{d_3-d_2}{d_2-d_1}(m_1-m_2)$$

$$m_3=\frac{m_2(d_3-d_1)-m_1(d_3-d_2)}{d_2-d_1}. \quad \textit{Ans.}$$

(b) Let M be the actual mass of the body and ρ is its density. Then its apparent mass is $\left(M-\frac{M}{\rho}\sigma\right)$ by the principle of Archemedes, where σ is the density of air.

Let D be the density of brass weight.

Then apparent mass of brass is $\left(W-\frac{W}{D}\sigma\right)$ where W is the mass of brass weight.

As the loads on the pans balance

$$M\left(1-\frac{\sigma}{\rho}\right)=W\left(1-\frac{\sigma}{D}\right)$$

or
$$M=W\left(1-\frac{\sigma}{D}\right)\left(1-\frac{\sigma}{\rho}\right)^{-1}.$$

Expanding and neglecting higher powers of $\frac{\sigma}{\rho}$

$$M=W\left(1-\frac{\sigma}{D}+\frac{\sigma}{\rho}\right).$$

When there is no air, $M=W$.

$$\therefore \quad \text{Error}=W-W\left(1-\frac{\sigma}{D}+\frac{\sigma}{\rho}\right)$$

$$\%\ \text{error}=\sigma\left(\frac{1}{D}-\frac{1}{\rho}\right)\times 100=\left(\frac{\sigma}{D}-\frac{\sigma}{\rho}\right)\times 100$$

$$=\left(\frac{1\cdot 2}{8000}-\frac{1\cdot 2}{12000}\right)\times 100=\cdot 005. \quad \textit{Ans.}$$

5. Let V = volume of the whole hydrometer.

Then $\frac{V}{10}$ = volume of the stem

$= \cdot 1 \times \alpha$ where α = area of cross-section or $$\alpha=\frac{V\times 10}{10}=V.$$

By law of floatation

$$(V-{\cdot}02\alpha)\times 1000g=W$$

where W = weight of the hydrometer

and $$(V-{\cdot}04\alpha)\times 1000\,sg=W$$

$$\therefore\quad (V-{\cdot}02V)\times 1000g=(V-{\cdot}04V)\times 1000\,sg \qquad (\because\ \alpha=V)$$

or $$s=1{\cdot}02.\ \textit{Ans.}$$

6. Considering 'vertical equilibrium' of the rod we have

$$T+F_b=12g+6g \text{ where } F_b=\text{buoyant force}$$

$$F_b=\frac{V}{2}\times 1000g \text{ where } V=\text{total volume of the rod}$$

$$\therefore\quad T+V\times 500g=18g.$$

Taking moment about A

$$F_b\times 3\times\cos\theta=12g\times 2\times\cos\theta+6g\times 4\times\cos\theta$$

$$F_b=16g$$

$$\therefore\quad \frac{V}{2}\times 1000g=16g$$

or $$V=32\times 10^{-3}\text{m}^3=32\times 10^3\text{ cc.}\quad \textit{Ans.}$$

$$T=18g-32\times 10^{-3}\times 500g=18g-16g=2g=19{\cdot}6\text{ N.}\quad \textit{Ans.}$$

7. Weight of the sphere $= W$ (say).

Weight of liquid displaced $=\frac{1}{2}\times\frac{4\pi}{3}\times{\cdot}1^3\times{\cdot}8\times 1000g.$

Weight of the second liquid displaced $=\frac{4\pi}{3}\times{\cdot}1^3\times\rho g.$

By law of floatation,

$$\frac{1}{2}\times\frac{4\pi}{3}\times{\cdot}1^3\times{\cdot}8\times 1000g=W=\frac{4\pi}{3}\times{\cdot}1^3\times\rho g$$

or $$\rho=\tfrac{1}{2}\times{\cdot}8\times 1000=400\text{ kg/m}^3.\quad \textit{Ans.}$$

8. Let h be the height of water which must be poured into the vessel. Mass of water poured $=\pi\times 2^2\times h\times 1=4\pi h$ gm.

Let h_{cm} be the height of the centre of mass of the system from the bottom.

$$h_{cm}=\frac{30\times 10+4\pi h\times h/2}{30+4\pi h}=\frac{300+2\pi h^2}{30+4\pi h}.$$

$$\left(\text{Formula, } x_{cm}=\frac{m_1x_1+m_2x_2}{m_1+m_2}\right)$$

$\therefore$ Length of the immersed portion $=2h_{cm}=\dfrac{300+2\pi h^2}{15+2\pi h}$.

Mass of the floating system $=(30+4\pi h)$.

Mass of displaced water $=\pi\times 2^2\times\left(\dfrac{300+2\pi h^2}{15+2\pi h}\right)\times 1$.

By law of floatation

$$30+4\pi h=4\pi\left(\frac{300+2\pi h^2}{15+2\pi h}\right)$$

$$(30+4\pi h)(15+2\pi h)=300\times 4\pi+8\pi^2h^2$$

or $$450+60\pi h+60\pi h+8\pi^2h^2=1200\pi+8\pi^2h^2$$

or $$120\pi h=1200\pi-450$$

or $$h=10-\frac{450}{120\pi}=10-1{\cdot}2=8{\cdot}8 \text{ cm}$$

$\therefore$ Mass of water $=\pi\times 2^2\times 8{\cdot}8\times 1=110{\cdot}6$ gm. *Ans.*

9. Let x be the height by which both the levels go up and V_2 is the volume of iron piece immersed in mercury. Then

$$V_2\,\rho_m g=V_0\,\rho_i g \text{ by law of floatation}$$

or $$V_2\,\rho_m=V_0\,\rho_i.$$

Suppose h be the initial height of the mercury columns in the vessels and v is the volume of the horizontal portion of the connecting tube. Since volume of mercury remains constant we have

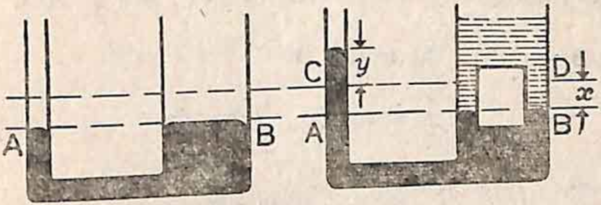
$$hA+hA'+v=(h+x)A+(h+x)A'-V_2+v$$

or $$V_2=(A+A')x$$

or $$x=\frac{V_0\rho_i}{(A+A')\rho_m}.\quad Ans.$$

10. Let AB be the initial level. When the iron piece is dropped into wider vessel, the level of mercury will rise by the same amount x (see sol. 9) and occupy the position CD. Let water be filled up to h to bring the mercury level to AB again. Now the iron piece will sink partly into the mercury and the rest of it will be in water. Let V_2 be

the volume of the portion submerged in mercury. Then obviously V_2 volume of mercury is displaced to the narrow vessel. Therefore, we can write $V_2=(x+y)A$ because the mercury transferred from the



wider vessel to the narrow vessel occupies $(x+y)$ length of it. Considering 'equality of pressure' on the initial level AB we have

$$(x+y)\rho_m g=h\rho_w g$$

where ρ_m = density of mercury, ρ_w = density of water

or
$$(x+y)\rho_m=h\rho_w$$

or
$$\frac{V_2}{A}\rho_m=h\rho_w \quad \text{or} \quad V_2=hA\frac{\rho_w}{\rho_m}.$$

From law of floatation

$$V_0\rho_i g=V_2\rho_m g+(V_0-V_2)\rho_w g$$

where ρ_i = density of iron

or
$$V_0\rho_i=V_2\rho_m+V_0\rho_w-V_2\rho_w=V_2(\rho_m-\rho_w)+V_0\rho_w$$

or
$$V_0\rho_i=hA\frac{\rho_w}{\rho_m}(\rho_m-\rho_w)+V_0\rho_w$$

or
$$hA\frac{\rho_w}{\rho_m}(\rho_m-\rho_w)=V_0(\rho_i-\rho_w)$$

or
$$h=\frac{\rho_m(\rho_i-\rho_w)V_0}{\rho_w(\rho_m-\rho_w)A}. \ Ans.$$

11. The velocity of efflux $=\sqrt{2gh}$.

Obviously when a particle falls through a vertical distance $(H-h)$, it moves through a horizontal distance x (say).

Then
$$x=(\sqrt{2gh})t \quad \text{and} \quad (H-h)=\tfrac{1}{2}gt^2$$

$$\therefore \quad H-h=\tfrac{1}{2}g.\frac{x^2}{2gh}=\frac{x^2}{4h}$$

$$x^2=4hH-4h^2=-(4h^2-4hH)=-(4h^2-4hH+H^2-H^2)$$
$$=H^2-(2h-H)^2.$$

Obviously x is maximum when $(2h-H)^2$ is minimum. It is a positive quantity. Its minimum value is zero.

$\therefore$ x is maximum when $2h-H=0$ or $h=\frac{H}{2}$. *Ans.*

12. By Bernoulli's theorem

$$\frac{1}{2}.50^2+\frac{p_1}{\rho}=\frac{1}{2}v^2+\frac{p_2}{\rho}$$

or
$$\frac{1}{2}(v^2-50^2)=\frac{p_1-p_2}{\rho}.$$

It is given that $p_1-p_2=100$ kgf/m^2 $=980$ N/m^2

$$\therefore\ \frac{1}{2}(v^2-50^2)=\frac{980}{1{\cdot}2} \quad \text{or} \quad v=64{\cdot}3 \text{ m/s. } \textit{Ans.}$$

13. The greatest possible value corresponds to the zero pressure over the wing. By Bernoulli's theorem

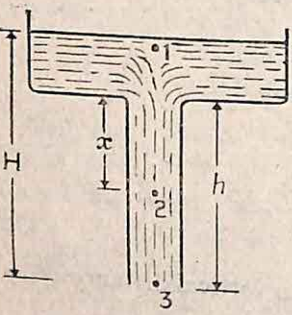
$$\frac{1}{2}v^2_{max}+0=0+\frac{p_0}{\rho} \text{ where } p_0=\text{atmospheric pressure}$$

or
$$v_{max}=\sqrt{\frac{2p_0}{\rho}}=\sqrt{\frac{2\times 1{\cdot}013\times 10^5}{1{\cdot}2}}$$

$(\because\ p_0=76 \text{ cm of } Hg=1{\cdot}013\times 10^5 \text{ N/m}^2)$

$=4{\cdot}1\times 10^2$ m/s. *Ans.*

14. Let us consider three points 1, 2, 3 in the flow of water. The positions of the points are as shown in the figure.



Applying Bernoulli's theorem to points 1, 2 and 3

$$\frac{p_0}{\rho}+\frac{1}{2}v_1^2+gH$$

$$=\frac{p_2}{\rho}+\frac{1}{2}v_2^2+g(h-x)$$

$$=\frac{p_0}{\rho}+\frac{1}{2}v_3^2+0.$$

By continuity equation $v_1A_1=A_2v_2=A_2v_3$.
Since $A_1>>A_2$, v_1 is negligible and $v_2=v_3=v$ (say).

$$\therefore \quad \frac{p_0}{\rho}+gH=\frac{p_2}{\rho}+\frac{1}{2}v^2+g(h-x)=\frac{p_0}{\rho}+\frac{1}{2}v^2$$

$$\therefore \quad v=\sqrt{2gH} \qquad \text{.. (i) } Ans.$$

and
$$\frac{p_0}{\rho}+gH=\frac{p_2}{\rho}+gH+g(h-x)$$

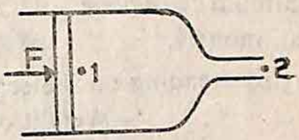
or
$$p_0=p_2+\rho g(h-x)$$

$$p_2=p_0-\rho g(h-x) \qquad \text{.. (ii) } Ans.$$

Thus pressure varies with distance from the upper end of the pipe according to the equation (ii) and velocity is a constant and is given by (i).

15. The pressure at the point 1

$$=\frac{F}{A}+p_0$$



where p_0 = atmospheric pressure.

Pressure at 2 = p_0.

Applying Bernoulli's theorem to 1 and 2

$$\frac{\frac{F}{A}+p_0}{\rho}+\frac{1}{2}v_1^2=\frac{p_0}{\rho}+\frac{1}{2}v_2^2$$

or
$$\frac{F}{A\rho}=\frac{1}{2}(v_2^2-v_1^2).$$

By 'continuity equation' $v_1A=v_2a$

$$\frac{F}{A\rho}=\frac{1}{2}\left(v_2^2-\frac{v_2^2a^2}{A^2}\right) \quad \text{or} \quad v_2^2=\frac{2F}{A\rho}\cdot\frac{1}{1-\frac{a^2}{A^2}}.$$

$$\because \quad a<<A, \quad \therefore \quad v_2^2=\frac{2F}{A\rho}$$

or
$$v_2=\sqrt{\frac{2F}{A\rho}}.\ Ans.$$

Solution to Test Questions

1. (i) When it is submerged more by a short distance, the air inside the balloon is compressed resulting in reduction of the buoyant force on it. Thus downward force (weight of the sinker attached + weight of air in the balloon) being in excess of the buoyant force it sinks to the bottom.

(ii) Let V' be the total volume of the block of wood and W is the weight of the block. Then by law of floatation

$$(V'-V)\times \rho_t g = W \text{ where } \rho_t = \text{density of water at } t°\text{C}.$$

Since wood has negligible coefficient of expansion V' may be considered constant.

$$\therefore \quad V = V' - \frac{W}{\rho_t g}.$$

As the temperature is gradually increased from 0°C, ρ_t increases and so V increases up to 4°C when density of water becomes maximum. Above 4°C, ρ_t decreases and so V decreases continuously.

(iii) Reading on the scale

$$= \text{Weight of liquid and beaker} + \text{buoyant force}$$
$$= W + V\rho_t g$$

where W = weight of liquid + beaker,
V = volume of the solid at t°C
and ρ_t = density of liquid at t°C

or
$$\text{Reading} = W + V_0(1+\gamma t)\times \frac{\rho_0}{1+\gamma' t} g$$

where V_0 = volume of the solid at 0°C
γ = coefficient of expansion of the solid
γ' = coefficient of expansion of the liquid

or
$$\text{Reading} = W + V_0\rho_0\{1-(\gamma'-\gamma)t\}g.$$

As t is increased, $\{1-(\gamma'-\gamma)t\}$ decreases and so *reading also decreases because* W is a constant.

(iv) In air
$$W_g - F_g = W_b - F_b$$

where W_g = weight of glass bulb, F_g = buoyant force on the bulb, W_b = weight of brass weights, F_b = buoyant force on the brass weights.

Since the volume of the bulb is greater than brass weights

$$F_b < F_g \quad \text{or} \quad W_b - W_g + F_g < F_g \quad \text{or} \quad W_b < W_g.$$

Thus when air is removed the bulb will become heavier than the brass weights.

(v) Let H be the height of liquid in the beaker and H' be the height of the liquid after all the ice melts.

Let A be the area of cross-section of the beaker.

Let V = volume of the ice block and V' is the volume of it inside the liquid.

Then $$V\rho_{ice}\times g = V'\times 1{\cdot}2\times 1000\times g$$

or $$V' = \frac{V}{1200}\rho_{ice}$$

Volume of liquid in the beaker $= AH - V'$

$$= AH - \frac{V\rho_{ice}}{1200}.$$

When all the ice melts, it is converted into water of the same mass but of volume, say, V''.

$$\therefore \quad V''\times 1000 = V\rho_{ice} \quad \text{or} \quad V'' = \frac{V\rho_{ice}}{1000}$$

$\therefore$ Total volume of liquid after ice melts

$$= \left(AH - \frac{V\rho_{ice}}{1200}\right) + \frac{V\rho_{ice}}{1000} = AH'$$

or $$A(H'-H) = V\rho_{ice}\left(\tfrac{1}{1000} - \tfrac{1}{1200}\right) = +\text{ve}.$$

$$\therefore \quad H' > H.$$

So the level of the liquid will rise.

(vi) Let h_1 and h_2 be the heights above the horizontal line.

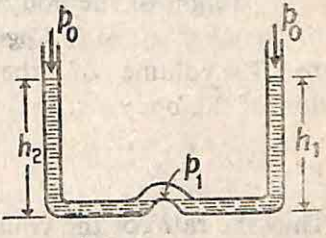
Let p_1 be the pressure of the entrapped air.

Then $p_1 - p_0 = \rho g h_2$

and also $p_1 - p_0 = \rho g h_1$

$\therefore \quad \rho g h_1 = \rho g h_2$

or $\quad h_1 = h_2$.



So the entrapped air will not affect the reading from one end to the other.

(vii) Steam is highly compressible. When a steam boiler blows up, steam expands violently causing considerable damage while water being incompressible expands nominally on being released of the pressure. So it does not cause much damage.

(viii) Since they are of the same radius, the buoyant force on each of them is the same.

$$\therefore \quad W - F_b = \frac{W}{g}\times a$$

where W is the weight of the hollow sphere and $$W'-F_b=\frac{W'}{g}\times a'$$

where W' = weight of solid sphere.

$$\therefore \quad W\left(1-\frac{a}{g}\right)=F_b=W'\left(1-\frac{a'}{g}\right).$$

But $$W'>W$$

$$\therefore \quad 1-\frac{a}{g}>1-\frac{a'}{g}$$

or $$a<a'.$$

Hence the solid sphere arrives at the ground earlier.

(ix) No. Rain drops of different sizes reach the earth with different speeds. The terminal velocity acquired by a drop is proportional to the square of its radius.

At the final stage,

upward viscous force = downward driving force

or $$6\pi\eta rv=\frac{4\pi}{3}r^3(\rho-\sigma)g$$

or $$v\propto r^2.$$

(x) For floatation we have

weight of the body = weight of the liquid displaced

$$V\rho g=V'\rho' g$$

where V = volume of the body, V' = volume of the submerged portion of the body

or $$\frac{V'}{V}=\frac{\rho}{\rho'}.$$

Thus the ratio of the volume submerged to the total volume is independent of the acceleration due to gravity.

Hence in free space and on the surface of Jupiter the cork will niether sink more nor float out more.

(xi) The pressure difference on the two sides of a tennis ball due to spin motion imparted to it is greater because of its roughness than a base ball. This is why it is easier to throw a curve with a tennis ball.

(xii) Due to high speed of the train, the pressure of the air between the train and the man may fall considerably and the pressure difference may be sufficiently high to push the man towards the speedy train.

(xiii) The level of water rises. See proof in Introductory Physics, Part I, Chapter 13.

(xiv) To pass on to the stable equilibrium position.

(xv) $p = p_0$ (atmospheric pressure). In general $p = p_0 + \rho(g+a)h$. Here $a = -g$. Therefore. $p = p_0$.

2. Let $x = BC$ = length submerged in water.

Then F_b = buoyant force $= x\alpha \times 1000g$

W = weight of the rod $= l\alpha = 1000sg$

where α is the area of cross-section of the rod.

Considering linear equilibrium of the rod

$$l\alpha \times 1000sg = x\alpha \times 1000g + R$$

where R is the reaction of the support at O.

Considering rotational equilibrium of the rod about O

$$l\alpha \times 1000sg \times \left(\frac{l}{2} - a\right)\cos\theta \equiv (x\alpha \times 1000g)\left(l - a - \frac{x}{2}\right)\cos\theta$$

or $ls\left(\frac{l}{2} - a\right) = x\left(l - a - \frac{x}{2}\right)$; or $ls(l-2a) = x(2l - 2a - x)$

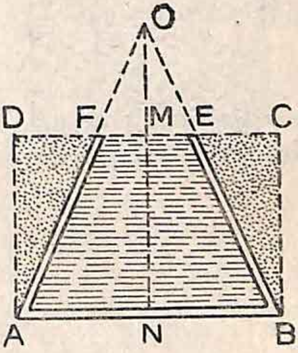
or $ls(l-2a) = 2x(l-a) - x^2$; or $x^2 - 2x(l-a) + ls(l-2a) = 0$

$$x = (l-a) \pm \sqrt{(l-a)^2 - ls(l-2a)}$$

Since $x < l - a$, only one solution is valid.

$$\therefore \quad x = l-a) - \sqrt{(l-a)^2 - ls(l-2a)}. \quad \textit{Ans.}$$

3. The thrust on the base is always equal to the 'area of the base × pressure on the base' i.e. it is equal to the weight of the vertical column of the liquid ($ABCD$) standing on the base AB, whatever be the shape of the vessel. Therefore here the total thrust on the base = weight of the liquid in the truncated cone $ABEF$ + weight of the liquid in the remaining portion of the liquid column $ABCD$.



This latter is the thrust exerted by the inclined walls AF and BE on the base. By Newton's third law the same thrust is exerted by the liquid on the inclined walls. Hence when this thrust is equal to the weight of the vessel, it will be just lifted. Therefore

W = weight of the liquid in the portion ($ABCD - ABEF$)

Volume of $ABCD = \pi R^2 h$

Volume of $ABEF$

$= $ volume of the cone $OAB -$ volume of the cone OEF

$= \frac{1}{3}\pi R^2 h_1 - \frac{1}{3}\pi r^2 h_2$ where $h_1 = ON$ and $h_2 = OM$.

Now $h_1 = R \cot\alpha$ and $h_2 = r \cot\alpha$

$\therefore \quad h_1 - h_2 = (R-r)\cot\alpha$

or $h = (R-r)\cot\alpha$ or $r = R - h\tan\alpha$

$\therefore$ Volume of $ABEF$

$$= \tfrac{1}{3}\pi R^2 R \cot\alpha - \tfrac{1}{3}\pi(R-h\tan\alpha)^2.(R-h\tan\alpha)\cot\alpha$$
$$= \tfrac{1}{3}\pi \cot\alpha[R^3-(R-h\tan\alpha)^3]$$

$\therefore$ Volume of the remaining

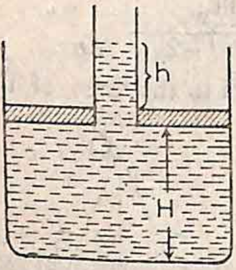
$$= \pi R^2 h - \tfrac{1}{3}\pi \cot\alpha[R^3 - R^3 + 3R^2 h\tan\alpha - 3Rh^2\tan^2\alpha + h^3\tan^3\alpha]$$
$$= \pi R^2 h - \pi[R^2 h - Rh^2\tan\alpha + \tfrac{1}{3}h^3\tan^2\alpha]$$
$$= \pi[R^2 h - R^2 h + Rh^2\tan\alpha - \tfrac{1}{3}h^3\tan^2\alpha]$$
$$= \pi[Rh^2\tan\alpha - \tfrac{1}{3}h^3\tan^2\alpha] = \pi h^2[R\tan\alpha - \tfrac{1}{3}h\tan^2\alpha]$$

Weight of the liquid in the remaining volume

$$= \pi h^2 \rho\tan\alpha(R - \tfrac{1}{3}h\tan\alpha)g$$

$$\therefore \quad W = \pi h^2 \rho\tan\alpha(R - \tfrac{1}{3}h\tan\alpha)g$$

or
$$\rho = \frac{W}{\pi h^2 g\tan\alpha(R - \frac{1}{3}h\tan\alpha)}. \quad \textit{Ans.}$$



4. Area of the piston $= \pi(R^2 - r^2)$.

Pressure on the base $= \rho g(H+h)$

where $\rho =$ density of water.

Also pressure on the base

$$= \frac{(W+m)g}{\pi R^2}$$

$$\left(\text{pressure} = \frac{\text{thrust}}{\text{area}}\right)$$

$$\therefore \quad \rho g(H+h) = \frac{(W+m)g}{\pi R^2}$$

or
$$H + h = \frac{(W+m)}{\pi R^2 \rho} \quad \cdot\cdot \quad (1)$$

Considering equilibrium of the piston

upward thrust $=$ weight of the piston

$$\pi(R^2 - r^2)\rho g h = Wg$$

$$h = \frac{W}{\pi(R^2 - r^2)\rho}. \quad \cdot\cdot \quad (2)$$

From (1) and (2)

$$H=\frac{W+m}{\pi R^2\rho}-\frac{W}{\pi(R^2-r^2)\rho}$$

$$=\frac{1}{\pi R^2\rho}\left(m-W\frac{r^2}{R^2-r^2}\right)$$

$$=\frac{1}{\pi\times 4^2\times 10^{-4}\times 1000}\left(\cdot 7-\frac{3\times 1^2\times 10^{-4}}{(4^2-1^2)\times 10^{-4}}\right)$$

$$=\frac{1}{1\cdot 6\pi}\left(\cdot 7-\frac{3}{15}\right)$$

$$H=\frac{\cdot 5}{1\cdot 6\pi}=\cdot 10 \text{ m}=10 \text{ cm}. \quad \textit{Ans.}$$

5. Take x-axis along the flow and y-axis perp. to it.

$$\vec{v}_{initial}=v\,\vec{i}$$

$$\vec{v}_{final}=v\,\vec{j}$$

$$\therefore \quad \Delta\vec{v}=v\,\vec{j}-v\,\vec{i}$$

$$\therefore \quad |\Delta\vec{v}\,|=\sqrt{v^2+v^2}=v\sqrt{2}.$$

The mass flowing through a section in every second $=\alpha v\rho$ kg/s.

$\therefore$ Rate of change of momentum per second

$$=v\sqrt{2}\times\alpha v\rho=\sqrt{2}\,\alpha v^2\rho.$$

By Newton's law this is force exerted by the gas on the pipe.

$\therefore$ Force required $=\sqrt{2}\,\alpha v^2\rho$. *Ans.*

6. Let V be the volume of the balloon and ρ_0 density of air. Then by law of floatation

$$V\rho_0 g=\tfrac{2}{3}V\rho_w g \quad \text{or} \quad \rho_0=\tfrac{2}{3}\rho_w \qquad \text{.. (i).}$$

The balloon will just sink when the *density of air becomes equal to the density of water due to compression*. Let V' be the compressed volume of the balloon.

Then $\dfrac{V\rho_0}{V'}=$ density of compressed air $=\rho_w$

or $$V'=\tfrac{2}{3}V.$$

By Boyle's law, $$p_0V=(p_0+\rho_w gH)V'$$

$$\rho_w g\times 10V=(\rho_w g10+\rho_w gH)\times\tfrac{2}{3}V$$

$$\frac{10\times 3}{2}=10+H \quad \text{or} \quad H=5 \text{ m}. \quad \textit{Ans.}$$

LESSON 9

Topics : **Surface Tension**
Errors and their Measurements

9.1 Surface Tension and Surface Energy

Surface Tension. The property by which a liquid surface tends to contract and occupy minimum area is called surface tension of the liquid. It is caused by forces of attraction between the molecules of the liquid. A molecule on the free surface of a liquid experiences a net resultant force which tends to draw it into the liquid. Surface tension is actually a manifestation of such forces experienced by the surface molecules.

If an imaginary line is drawn on a liquid surface, then the force acting per unit length of this line is defined as the surface tension. Its unit is therefore newton/metre. This force acts along the liquid surface. For curved surfaces, the force acts tangentially to the liquid surface at every point.

Surface Energy. Due to surface tension a liquid surface possesses potential energy. This energy per unit area of the surface is called the surace energy of the liquid. Unit, joule per square metre. The surface energy of a liquid has the same numerical value as its surface tension. Surface tension of a liquid depends on temperature. It decreases with rise of temperature.

Excess of Pressure. Inside a soap bubble or a gas bubble inside a liquid, there must be pressure in excess of the outside pressure to balance the tendency of the liquid surface to contract due to surface tension.

$$p \text{ (excess of pressure)} = T\left(\frac{1}{r_1}+\frac{1}{r_2}\right) \text{ in general,}$$

where T is the surface tension of the liquid, and r_1 and r_2 are the principal radii of curvature of the bubble in two mutually perpendicular directions.

For a spherical soap bubble, $r_1=r_2=r$ and there are two free sufaces of the liquid.

$$\therefore \quad p=\frac{4T}{r}.$$

For a gas bubble inside a liquid, $r_1=r_2=r$ and there is only one surface.

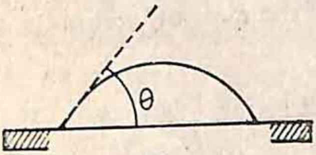
$$\therefore \quad p=\frac{2T}{r}.$$

For a cylindrical surface $r_1=r$ and $r_2=\infty$ and there are two surfaces.

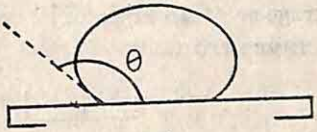
$$\therefore \quad p=\frac{2T}{r}.$$

Angle of Contact. The angle made by the surface of a liquid with the solid surface inside the liquid at the point of contact is called angle of contact. It is this angle at which the surface tension acts on the wall of the container.

The angle of contact θ depends on the natures of the liquid and solid in contact. If the liquid 'wets' the solid (e.g., water and glass), the angle of contact is zero. In most cases, θ is acute (fig. i). In the special case of mercury on glass, θ is obtuse (fig. ii).



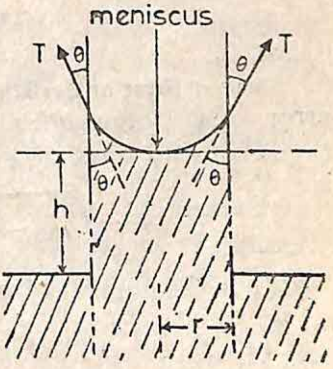
(i)



(ii)

Rise of a liquid in a capillary tube. In a thin (capillary) tube, the free surface of the liquid becomes curved. The forces of surface tension at the edges of the liquid surface then acquire a vertical component.

The upward force by which a liquid surface is pulled up in a capillary tube is $2\pi r\,T\cos\theta$, and downward force due to gravitational pull on the mass of liquid in the tube is $(\pi r^2h+v)\ \rho\, g$, where v is the volume above the liquid meniscus. If $\theta=0°$, the meniscus is hemispherical in shape. Then $v=$ difference of volumes of cylinder of radius r and height r and volume of hemisphere of radius r



$$=\pi r^3-\frac{2}{3}\pi r^3=\frac{1}{3}\pi r^3.$$

When $\theta \neq 0$, we cannot calculate v which is generally very small and so it may be neglected.

For equilibrium we have

$$(\pi r^2 h + v)\rho g = 2\pi r T \cos\theta.$$

When a glass capillary tube is dipped in mercury, the meniscus is convex, since the angle of contact is obtuse. The surface tension forces now acquire a downward component, and the level of mercury inside the tube falls below the level outside it. The relation $2T \cos\theta = h\rho g r$ may be used to obtain the fall in the mercury level.

9.2 Errors and their Propagation

Errors. If $\overline{X}$ is the average value of a number of observations of a physical quantity then $X_1 - \overline{X}$, $X_2 - \overline{X}$, $X_3 - \overline{X}$ .... are called the errors denoted by δ_1, δ_2, δ_3......

Average or Mean Error. The average of the mod of errors is called the average error.

$$\overline{\delta} = \frac{|\delta_1| + |\delta_2| + |\delta_3| \ldots. |\delta_n|}{n}.$$

Mean Square Error. The mean of the squares of the errors is called mean square error.

$$\overline{\delta^2} = \frac{\delta_1^2 + \delta_2^2 + \ldots\ldots + \delta_n^2}{n}.$$

Root Mean Square Error. The square root of the mean square error is called root mean square (rms) error.

$$\delta_{rms} = \sqrt{\frac{\delta_1^2 + \delta_2^2 + \ldots. \delta_n^2}{n}}.$$

Standard Error or Deviation. The square root of the mean square error of a *large number* of observations (infinite number of observations) is called standard error or standard deviation.

$$\sigma = \sqrt{\frac{\delta_1^2 + \delta_2^2 + \ldots. \delta_n^2}{n}} \text{ when } n \text{ is large.}$$

If n is the number of observations, then the standard deviation (σ) is

$$\sigma = \sqrt{\frac{\delta_1^2 + \delta_2^2 + \ldots. \delta_n^2}{n-1}}.$$

Propagation rule of mean or average deviation or errors. Absolute errors propagate in a sum or difference of two physical quantities.

Percentage errors or deviation propagate in a product or quotient of two physical quantities.

Propagation Rules of Standard Deviations.

If $A = B + C$, then

$$\sigma_A^2 = \sigma_B^2 + \sigma_C^2.$$

If $A = B \times C$, then

$$\left(\frac{\sigma_A}{A}\right)^2 = \left(\frac{\sigma_B}{B}\right)^2 + \left(\frac{\sigma_C}{C}\right)^2.$$

If $A = B/C$, then

$$\left(\frac{\sigma_A}{A}\right)^2 = \left(\frac{\sigma_B}{B}\right)^2 + \left(\frac{\sigma_C}{C}\right)^2.$$

Significant figures of a number. Consider a number 37489210. This is quite a big number. All the figures are not equally significant here. For example figure 1 here stands for 10 which is negligible in comparison to figure 3 whose 'place value' is 3000000. The figures which are significant in a number are called significant figures. These are figures occurring in the beginning of a number. The above number is 37490000, 37500000, 38000000....up to 4th, 3rd, 2nd places of significant figure.

Rules for rounding of significant figures. If the succeeding figure is greater than 5, the figure is increased by 1, otherwise it is left unchanged. If the succeeding figure be 5 itself then the figure is raised by 1 if it is odd, and left unchanged if it is even.

CRITICAL QUESTIONS

1. The surface of a liquid is frequently likened to a stretched membrane. In what respect does the analogy fail and in what respect is it similar ?

Ans. When an elastic membrane is stretched, the force of tension depends on the amount of deformation of the membrane. But in case of the surface of a liquid it does not depend on the increase or decrease of area. It is determined by the property of the liquid. They are similar in respect of their tendency to contract and occupy minimum area.

2. To remove a grease spot from a fabric petrol should be applied to the edges of the spot, while the spot itself should never be wetted with petrol immediately. Why not ?

Ans. The surface tension of pure petrol is less than that of petrol in which grease is dissolved. For this reason petrol applied to the edges will contract towards the centre. If the spot itself is wetted, it will spread all over the fabric due to capillary action.

3. Water can rise to a height h in a certain capillary tube. Suppose that this tube is immersed in water so that only a height $h/2$ is above the surface. Will you have a fountain or not ? Explain.

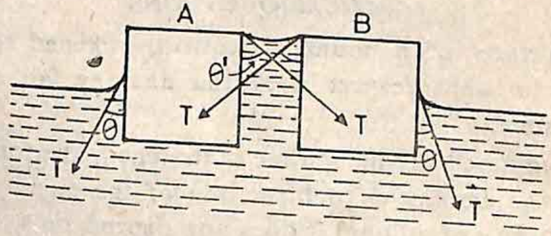
Ans. No, we will not have a fountain. The curvature of the free liquid surface at the top of the tube will change till the upward force $2\pi r\, T\cos\theta$ balances the gravitational pull of the column of liquid of height $h/2$.

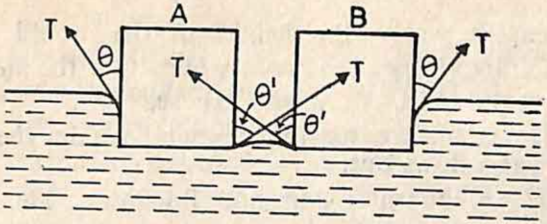
4. Why is moisture retained longer in soil if it is harrowed ?

Ans. When soil is not harrowed, there are large capillaries in the soil. Water in the soil rises up the capillary holes to the surface from where it is evaporated continuosly. Thus the soil continuously loses water. When harrowed, all these capillaries are destroyed, and so capillary suction stops altogether. Thus water is retained in the soil longer.

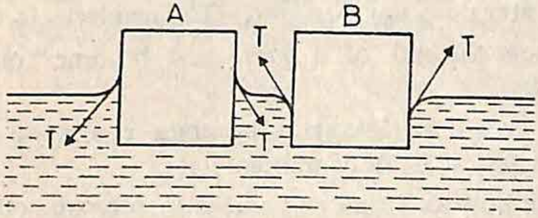
5. Light bodies that can be wetted by water (for example, two match box sticks) attract each other while floating. If they cannot be wetted (say when coated with thin layer of wax) they again attract each other. If, however, one can be wetted and the other cannot be wetted they repel each other. Explain these phenomena.

Ans. When both are wetted by water, the shape of the liquid surfaces are shown in figure. Resolution of forces show that the each body experiences force towards each other. Force on A to the right $= T(\sin\theta' - \sin\theta)$. Since the height of water is insufficient, $\theta' > \theta$ or $\sin\theta' > \sin\theta$. So the force is positive to the right. In terms pressure difference on the curved surfaces the same can be understood in this way : Because of curvature of the surface of the liquid between A and B, the pressure at any point is less than the atmospheric





pressure and so they are pressed together by the side atmospheric pressure.



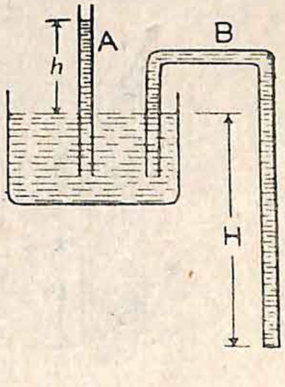
When both the bodies cannot be wetted by water, the shape and direction of forces are as shown in the second figure. Similar consideration show that they attract each other.

When one, say *A*, cannot be wetted and *B* can be wetted, then shape and inclination of forces are as shown in the third figure. Similar consideration show that they repel each other.

6. Why rain drops are spherical in shape ?

Ans. Because of surface tension of water. Due to surface tension the surface of a small volume of water tends to occupy minimum area. The area of a given volume of water has minimun value for spherical shape.

7. Two capillary tubes *A* and *B* are immersed in water—one is straight and the other is in form of a rectangular U-tube. The tube *A* is sufficiently long. The lower end of the bent tube is at a depth *H*. What form will the meniscus take and will therse be any flow of the water ? Consider the following five cases : (a) $H>h$, (b) $H=h$, (c) $0<H<h$, (d) $H=0$, (e) $H<0$.



Ans. In a capillary tube the height of the liquid that can be retained is given by $\pi r^2 h\rho g = 2\pi r T\cos\theta$ where θ is the angle at which the liquid meets the solid. $h\rho gr = 2T\cos\theta$.

(a) Since $H > h$, surface tension cannot sustain the column of water and so water flows out.

(b) Since $H = h$, the water does not flow out. The meniscus is *convex*.

(c) The water does not flow out. The meniscus is convex and is less curved than in the second case.

(d) The water does not flow out. The meniscus is flat.

(e) The water does not flow out. The meniscus is concave.

8 Why does the end of a glass rod become round on being heated strongly ?

Ans. The end of a glass rod becomes round on being heated because of surface tension of molten glass.

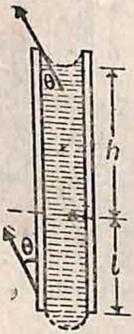
9. Explain how an iron can be used to remove greasy spots from clothing.

Ans. The grease melts, and capillary forces carry it to the surfrce of the cold fabric placed under the clothing, wher it is soaked.

10. Why do drops of water appear at the end of a piece of fire-wood when it is being dried in the sun while its other end is in the shadow ?

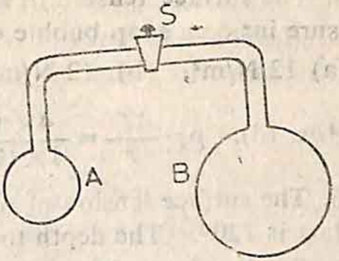
Ans. The end in the shadow is colder than the end in the sun. The capillary forces drive the water in this direction.

11. A capillary tube is dipped in water vertically. It is sufficiently long so that water rises to the maximum height 'h' in the tube. The length of the portion immersed in water is $l < h$. The lower end of the tube is closed, the tube is taken out and opened again. Will all the water flow out of the tube ? Explain.



Ans. No, no water will come out. When the tube is taken out, a convex meniscus is formed and a force due to surface tension is called into play in the upward direction to sustain the water in the tube. Thus the length of the water column remaining in the tube will be $l+h$.

12. Two soap bubbles A and B of different diameters are blown at the two ends of a bent tube. By opening the stop cock S, the two bubbles are put in communication. What will happen ?



Ans. The excess pressure inside a soap bubble is inversely proportional to its radius. Hence, the pressure inside A is greater than the pressure inside B. Air will flow from A to B. Therefore, A will become smaller and B will become larger.

OBJECTIVE QUESTIONS

1. When a capillary glass tube of insufficient length (i.e., length smaller than the capillary rise possible in that tube) is immersed in water, the water surface at the top of the tube makes an angle θ with the vertical wall of the tube. Then

(a) $\theta > \theta_c$, (b) $\theta < \theta_c$, (c) $\theta = \theta_c$, (d) 90°, where θ_c is the angle of contact.

Ans. (a).

2. On increasing the temperature of water its surface tension

(a) increases, (b) decreases, (c) remains unchanged, (d) non of these.

Ans. (b).

3. The excess of pressure inside a bubble in a liquid is

(a) $\frac{4T}{r}$, (b) $\frac{T}{r}$, (c) $\frac{2T}{r}$, (d) $\frac{3T}{r}$.

Ans. (c).

4. A bubble of radius 1 cm is blown in water. The work done by the blower is ($T = \cdot 07$ N/m and $\pi = 3$)

(a) 100 mJ, (b) 84 μJ, (c) 84 mJ, (d) 100 μJ.

Ans. (b). Work done = area × surface energy

$$= \text{area} \times \text{surface tension}$$
$$= 4\pi r^2 T = 4 \times 3 \times \cdot 01^2 \times \cdot 07$$
$$= 84 \times 10^{-6}\ \text{J} = 84\ \mu\text{J}.$$

5. The surface tension of soap water is ·03 N/m. The excess of pressure inside a soap bubble of radius 1 mm is

(a) 12 N/m², (b) ·12 N/m², (c) 1·2 N/m², (d) 120 N/m².

Ans. (d). $p=\frac{4T}{r}=\frac{4\times\cdot03}{1\times10^{-3}}=120\ \text{N/m}^2$.

6. The surface tension of mercury is ·50 N/m and its angle of contact is 120°. The depth to which it is depressed in a capillary tube of radius 1 mm is (neglect mass in the meniscus)

(a) ·375 cm, (b) 3·75 cm, (c) 3·75 mm, (d) ·375 mm.

Ans. (c). We have $2\pi rT\cos\theta=\pi r^2\rho hg$ (neglecting mass in the meniscus).

Hence $$h=\frac{2T\cos\theta}{\rho gr}=\frac{2\times\cdot5\times\cos120^\circ}{13600\times9\cdot8\times10^{-3}}$$
$$=3\cdot75\times10^{-3}\ \text{m}=3\cdot75\ \text{mm}.$$

7. A soap bubble is placed on another soap bubble. The angles at which the films meet at the points of contact is

(a) 90°, (b) 60°, (c) 120°, (d) 45°.

Ans. (c). At any point the forces of surface tension are in equilibrium and they are of the same magnitude. This is possible only when the angles between them are 120°.

8. The meniscus between two vertical parallel glass plates dipped in water is

(a) spherical, (b) flat, (c) cylindrical, (d) none of these.

Ans. (c).

9. A liquid wets a solid completely. The meniscus of the liquid in a sufficiently long tube is

(a) flat, (b) concave, (c) convex, (d) none of these.

Ans. (b).

10. Water rises in a capillary tube completely wettable by water up to a height h. The tube is gradually depressed into the water. The height at which the surface becomes flat is

(a) $\frac{1}{2}h$, (b) $\frac{3}{4}h$, (c) $\frac{1}{4}h$, (d) zero.

Ans. (d). For equilibrium in an insufficiently long tube we have,

$$2\pi rT\cos\theta=(\pi r^2h\rho)g \quad\text{or}\quad \cos\theta=\frac{hr\rho g}{2T}.$$

When the surface is flat $\theta=90^\circ$ $\therefore$ $h=0$.

11. Water rises in a capillary tube completely wettable by water to a height h. The tube is gradually depressed into the water. When the height above the water level is $\frac{1}{2}h$, the radius of curvature of the meniscus is (the radius of the capillary is r)

(a) r, (b) $2r$, (c) $\frac{3}{4}r$, (d) $\frac{1}{4}r$.

Ans. (b). For equilibrium of liquid we have

$$2\pi rT\cos\theta=(\pi r^2h'\rho)g \quad \text{or} \quad 2T\cos\theta=rh'\rho g.$$

When sufficiently long $\theta=0°$ and $h'=h$.

$$\therefore \quad 2T=rh\rho g.$$

When $h'=\dfrac{h}{2}$, $2T\cos\theta=r.\dfrac{h}{2}\rho g$ or $\cos\theta=\frac{1}{2}$.

If R is the radius of the meniscus, then

$$r=R\cos\theta$$

$$\text{or} \quad R=\frac{r}{\cos\theta}=2r.$$

Errors

1. When the number 6498734 is rounded up to four significant figures, it becomes....

Ans. 6499000 or $6{\cdot}499\times10^7$.

2. When the number 6·03587 is rounded up to second place of decimals, it becomes....

Ans. 6·04.

3. When the number 5·678536 is rounded up to third place of decimals, it becomes.....

Ans. 5·678.

4. When 2·0347 is added to 15·7, the sum is.....

Ans. The ordinary sum is 17·7347. Since there are three significant figures in the minimum, the ordinary sum has to be rounded off up to three places. On rounding 17·7347 up to three places it becomes 17·7. So the answer is 17·7.

5. In the measurement of n from the formula

$$n=\frac{2Wgl}{\pi r^4\theta}$$

the quantity which should be measured with the best care is....

Ans. r.

6. The measure of the length of a rectangle are :

$l=(3{\cdot}00 \pm 0{\cdot}01)$ cm and breadth $b=(2{\cdot}00 \pm 0{\cdot}02$ cm).

The area of the rectangle is. . . . .

Ans. $l=3{\cdot}00 \pm 0{\cdot}33\%,\ b=2{\cdot}00 \pm 1\%$

$A=l \times b=6{\cdot}00 \pm 1{\cdot}33\%=(6{\cdot}00 \pm 0{\cdot}08)$ cm². *Ans.*

7. Time of fall of a body $=(10{\cdot}0 \pm 00{\cdot}5)$ s.

Acceleration due to gravity $=(9{\cdot}8 \pm 0{\cdot}2)$ m/s².

The percentage error in distance described is. . . .

Ans.
$$S=\frac{1}{2}gt^2=b\times 10{\cdot}0\times 9{\cdot}8=49$$

$$f=10{\cdot}0 \pm 00{\cdot}5=10{\cdot}0 \pm 5{\cdot}0\%$$
$$g=9{\cdot}8 \pm 0{\cdot}2=9{\cdot}8 \pm 2{\cdot}0\%$$

$$\partial S=\frac{1}{2}g\,2t\partial t+\frac{1}{2}t^2\partial g$$

$$\therefore \quad \frac{\partial S}{S}=2\frac{\partial f}{t}+\frac{\partial g}{g}$$

$$\%\text{ error in } S=2\times 5{\cdot}0+2{\cdot}0$$
$$=10{\cdot}0+2{\cdot}0$$
$$=12$$
$$\therefore \quad S=49 \pm 12\%=(49 \pm 0{\cdot}6)\text{ cm.} \quad \textit{Ans.}$$

EXAMPLES

1. A bubble of soap water is increased in size from 2 cm to 4 cm. Calculate the work done by the blower.

Surface tension of soap water $=\cdot 03$ N/m.

Sol. Work done = increase in area × surface tension

$=2\times 4\pi\,(r_2^2-r_1^2)\times T$, multiplied by 2 because there are two surfaces of a soap bubble

$=8\pi\,(4^2-2^2)\times 10^{-4}$ joule $=9{\cdot}0\times 18^{-4}$ joule. *Ans.*

2. The end of a capillary tube with a radius r is immersed into water. Is mechanical energy conserved when the water rises in the tube ? The tube is sufficiently long. If not, calculate the energy change.

Sol. We have in the equilibrium position ($\theta=0°$ for pure water and glass)

$$2\pi\, rT\cos 0°=\pi r^2 h\rho g \quad \text{or} \quad h=\frac{2T}{\rho g r}.$$

Work done by surface tension

$$=(2\pi rT)\times h=\frac{4\pi T^2}{\rho g}.$$

Potential energy of water in the tube $=(\pi r^2 h\rho)\,gh/2$, it is multiplied by $\frac{1}{2}h$ because the cg of the water in the capillary tube is at a height $\frac{1}{2}h$

$$=\frac{2\pi T^2}{\rho g}.$$

Thus it is seen that the mechanical energy is not conserved.

$$\therefore \text{ Mechanical energy loss}=\frac{4\pi T^2}{\rho g}-\frac{2\pi T^2}{\rho g}=\frac{2\pi T^2}{\rho g}.\ \textit{Ans.}$$

This energy is converted into heat.

3. A cube of mass $m=800$ gm wettable by water floats on the surface of water. The cube is 10 cm on each edge. By what additional distance is it buoyed up or down by surface tension ?

Surface tension of water $=\cdot07$ N/m.

Sol. If surface tension is neglected the condition for floating gives

$$800\times10^{-3}g=(\cdot1^2\times x\rho)g$$

or $$x=\cdot08\text{ m}\ \ (\rho=1000\ kg/m^3 \text{ for water})$$

Since the cube is wettable by water, angle of contact is zero and the force of surface tension acts vertically downwards. So it is buoyed down by surface tension.

$$\therefore\ \ 800\times10^{-3}g+4\times\cdot1\times\cdot07=(\cdot1^2\times x'\rho)g$$

or $$x=\cdot08+\frac{\cdot028}{98}=\cdot08+2\cdot8\times10^{-4}$$

$$\therefore\ \text{The additional distance}=2\cdot8\times10^{-4}\text{ m.}$$

4. A glass capillary tube of internal radius $r=0\cdot25$ mm is immersed into water. The top end of the tube projects by 2 cm above the surface of the water. At what angle does the liquid meet the tube ?

Surface tension of water $=\cdot07$ N/m.

Sol. Water wets glass and so the angle of contact is zero. Neglecting the small mass is the meniscus we have for full rise,

$$2\pi rT=\pi r^2 h\rho g\ \text{ or }\ h=\frac{2T}{r\rho g}=\frac{2\times\cdot07}{\cdot25\times10^{-3}\times1000\times9\cdot8}$$

$$=\cdot057\text{ m}=5\cdot7\text{ cm.}$$

But here the tube is only 2 cm above water and so water will rise by 2 cm and meet the tube at such an angle that

$$2\pi rT\cos\theta=\pi r^2 h'\rho g$$

or $$2T\cos\theta=h'r\rho g \quad \text{or} \quad \cos\theta=\frac{h'r\rho g}{2T}.$$

$$\therefore \quad \cos\theta=\frac{2\times10^{-2}\times\cdot25\times10^{-3}\times1000\times9{\cdot}8}{2\times\cdot07}=\cdot35$$

or $\theta=70°$. *Ans.*

5. A soap bubble of radius r is placed on another soap bubble of radius R. What is the radius of the film separating the two bubbles ?

Sol. Excess of pressure inside the first bubble $=\dfrac{4T}{r}$

Excess of pressure in the second $=\dfrac{4T}{R}$.

$\therefore$ Excess of pressure on the two sides of separating film

$$=4T\left(\frac{1}{r}-\frac{1}{R}\right).$$

If R' is the radius of the film, then excess of pressure inside of it is

$$\frac{4T}{R'}.$$

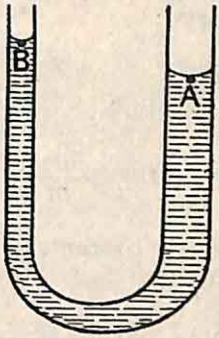
$$\therefore \quad \frac{4T}{R'}=4T\left(\frac{1}{r}-\frac{1}{R}\right)$$

or $$\frac{1}{R'}=\frac{1}{r}-\frac{1}{R} \quad \text{or} \quad R'=\frac{Rr}{R-r}.$$ *Ans.*

6. Calculate the difference in water levels in two communicating capillary tubes of diameter $d=1$ mm and $d=1{\cdot}5$ mm. Surface tension of water $=\cdot07$ N/m and angle of contact between glass and water $=0°$.

Sol. Pressure at $A=p_0-\dfrac{2T\cos\theta}{r_2}$

($\because$ pressure inside a curved surface is greater than outside).



Pressure at $B = p_0 - \frac{2T\cos\theta}{r_1}$

$\therefore$ Pressure difference

$$= 2T\cos\theta\left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

Let this pressure difference correspond to '*h*' unit of the liquid.

Then $2T\cos\theta\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = \rho g h$

$$h = \frac{2T\cos\theta}{\rho g}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

$$= \frac{2\times{\cdot}07}{1000\times 9{\cdot}8}\left(\frac{1}{1\times 10^{-3}} - \frac{1}{1{\cdot}5\times 10^{-3}}\right)$$

$$= 4{\cdot}76\times 10^{-3}\text{ m} = 4{\cdot}76\text{ mm.} \quad \textit{Ans.}$$

ERRORS

1. The errors in ten observations of a physical quantity are $-0{\cdot}55$, $+0{\cdot}43$, $+0{\cdot}05$, $-0{\cdot}36$, $0{\cdot}08$, $-0{\cdot}05$, $0{\cdot}00$, $-0{\cdot}03$, $-0{\cdot}05$, $+0{\cdot}06$. Calculate the mean deviation, root mean square deviation and standard deviation.

Sol. $$\delta = \frac{0{\cdot}55+0{\cdot}43+0{\cdot}05+0{\cdot}36+0{\cdot}08+0{\cdot}05+0{\cdot}00+0{\cdot}03+0{\cdot}05+0{\cdot}06}{10}$$

$$= 0{\cdot}165 = 0{\cdot}16. \quad \textit{Ans.}$$

$$\delta^2_{rms} = \frac{0{\cdot}55^2+0{\cdot}43^2+0{\cdot}05^2+0{\cdot}36^2+0{\cdot}08^2+0{\cdot}05^2+0{\cdot}00^2+0{\cdot}03^2+0{\cdot}05^2+0{\cdot}06^2}{10}$$

$$= \frac{{\cdot}6354}{10} = {\cdot}06354$$

or $$\delta_{rms} = 0{\cdot}2520 = 0{\cdot}25. \quad \textit{Ans.}$$

$$\sigma \text{ (standard deviation)} = \sqrt{\frac{{\cdot}6354}{9}} = 0{\cdot}3254 = 0{\cdot}32. \quad \textit{Ans.}$$

2. Show that percentage error in the measurement of a physical quantity m given by $m=k\tan\theta$ where k is a constant is minimum when $\theta=45°$.

Sol. Since $m=k\tan\theta$,

differentiating, $\partial m=k\sec^2\theta\,\partial\theta$

$$\therefore \quad \%\text{ error}=\frac{\partial m}{m}\times 100=\frac{2k}{\sin 2\theta}\times 100.$$

Obviously percentage error is minimum when $\sin 2\theta=1$

or $\theta=45°$.

3. Show that the percentage error in the measurement of a physical quantity X given by $X=\frac{al}{l_0-l}$ where a and l_0 are constants is minimum when $l=l_0/2$.

Sol. We have $X=\frac{al}{l_0-l}$.

Differentiating, $\partial X=\frac{a\partial l}{l_0-l}+al(-1)(l_0-l)^{-2}(-\partial l)$

$$=\frac{a\partial l}{l_0-l}+\frac{al\partial l}{(l_0-l)^2}.$$

$$\frac{\partial X}{X}=\frac{\partial l}{l}+\frac{\partial l}{l_0-l}=\frac{l_0\partial l}{l(l_0-l)}$$

Obviously $\frac{\partial X}{X}$ is minimum when $l(l_0-l)$ is maximum,

that is, $\frac{d}{dl}(l_0l-l^2)=0$ or $l_0-2l=0$

or $l=\frac{l_0}{2}$. Proved.

EXERCISES

1. A capillary tube is dipped in water with its lower end =10 cm below the surface of water. Water rises in the tube to a height of $h=4$ cm. What gauge pressure (i.e., excess of pressure

above the atmospheric pressure; a gauge measures this difference) is required to blow a hemispherical bubble at the lower end of the tube ?

2. Derive an expression for the height of capillary rise between two parallel plates dipping in a liquid of density σ separated by a distance d. The surface tension of the liquid is T.

3. A tube of circular section and outer radius 0·14 cm is closed at one end. This end is weighted so that the tube floats vertically. The total mass of the tube is 0·20 gm. If the angle of contact is zero, how far below the water surface is the bottom of the tube ? Surface tension of water = ·07 N/m.

4. A drop of water of mass $m = 0{\cdot}2$ gm is placed between two well cleaned glass plates, the distance between which is 0·01 cm. Find the force of attraction between the plates. Surface tension of water = ·07 N/m.

5. There are 1000 droplets of mercury of 1 mm diameter on a glass plate. Subsequently they merged into one big drop. How will the surface energy of the surface layer change ? The process is isothermal. Surface tension of mercury = ·475 N/m.

6. A vessel filled with air under pressure p_0 contains a soap bubble of diameter d. The air pressure having been reduced isothermally n-fold, the bubble diameter increased r-fold. Find the surface tension of the soap water solution.

7. The diameter of a gas bubble formed at the bottom of a pond is $d = 4$ cm. When the bubble rises to the surface its diameter increases $n = 1{\cdot}1$ times. Find the depth of the pond.

8. Find the difference in height of mercury columns in two communicating vertical capillary tubes whose diameters are $d_1 = 0{\cdot}5$ mm, and $d_2 = 1{\cdot}0$ mm, if the angle of contact is $\theta = 138°$. Surface tension of mercury = ·475 N/m and relative density of mercury = 13·6.

9. A glass rod of diameter $d_1 = 1{\cdot}5$ mm is inserted symmetrically into a glass capillary with inside diameter $d_2 = 2{\cdot}0$ mm. Then the whole arrangement is vertically oriented and brought in contact with the surface of water. To what height will the water rise in the capillary ? Surface tension of water = 0·07 N/m.

TEST YOURSELF

1. (i) A liquid always meets a solid surface at the angle of contact. Is it true or false ?

(ii) Water rises in a capillary tube up to h when it is sufficiently long. If the tube is then depressed to an extent $l > h$ and then lifted out of water after closing the lower end, what length of water will be retained in the tube on opening the lower end in air ?

(iii) When there is a thin layer of water in between two glass plates there is a strong attraction between them. Explain this. Do the plates attract when there is a thin layer of mercury ?

(iv) Is a floating glass cube in mercury buoyed up or down by its surface tension in water ?

(v) How many oscillations should be tried at 50 cm length of a simple pendulum if the least count of a stop watch is 1 s and the least count of a metre scale is 1 mm ?

(vi) The force of attraction between identical molecules is called ....force and that between dissimilar molecules is called...... force.

(vii) The excess of pressure inside a soap bubble of diameter d is

(a) $\frac{5T}{d}$, (b) $\frac{8T}{d}$, (c) $\frac{2T}{d}$, (d) $\frac{T}{d}$.

(viii) The standard deviation of n deviations or errors is

(a) $\sqrt{\frac{\Sigma|\delta|^2}{n}}$, (b) $\sqrt{\frac{\Sigma|\delta|^2}{n+1}}$, (c) $\sqrt{\frac{\Sigma|\delta|^2}{n-1}}$, (d) $\sqrt{\frac{\Sigma|\delta|^2}{n-2}}$.

(ix) The root mean square deviation of n deviations is

(a) $\sqrt{\frac{\Sigma|\delta|^2}{n}}$, (b) $\sqrt{\frac{\Sigma|\delta|^2}{n+1}}$, (c) $\sqrt{\frac{\Sigma|\delta|^2}{n-1}}$, (d) $\sqrt{\frac{\Sigma|\delta|^2}{n-1}}$.

2× 10 = 20

Answer any five of the following : 5× 6 = 30

2. A glass tube of inside diameter 1 mm is dipped verticlly into a container of mercury, with its lower end 1 cm below the mercury surface. What is the gauge pressure of air in the tube to blow a hemispherical bubble at its lower end ? Surface tension of mercury = ·500 N/m.

3. A tube of insufficient length is immersed in water (surface tension = 0·07 N/m) with 3 cm of it projecting vertically upwards outside the water. What is the radius of the meniscus ? Radius of the tube = 1 mm.

4. Find the pressure in an air bubble of diameter $d = 4\ \mu\text{m}$, located at a depth $h = 5$ m. The atmospheric pressure $p_0 = 10^5\ \text{N/m}^2$. Surface tension of water = ·07 N/m.

5. Find the force of attraction between two parallel glass plates, separated by distance $d = 0{\cdot}1$ mm, after a drop of water of mass $m = 70$ mg was introduced between them. The wetting is assumed to be complete. Surface tension of water = 0·07 N/m.

6. A vertical capillary is brought in contact with the water surface. What amount of heat is liberated while the water rises along the capillary? The wetting is assumed to be complete, the surface tension equals to T.

7. The measure of the radius of a sphere is (4·22 ± 2%) cm. Calculate the percentage error in volume and area of the sphere.

Answers of Exercises

1. 1372 N/m². 2. $\frac{2T}{\rho g d}$. 3. 4·2 cm. 4. 2·8 N. 5. $1{\cdot}34 \times 10^{-5}$ J.

6. $T = \frac{1}{8} p_0 d \, \dfrac{1 - \frac{r^3}{n}}{r^2 - 1}$. 7. 5 m. 8. 10·6 mm. 9. 5·7 cm.

Answers to Test Questions

1. See solutions. 2. 3333 N/m². 3. 1·4 mm. 4. $2{\cdot}2 \times 10^5\ \text{N/m}^2$.

5. ·98 N. 6. $\frac{2\pi T^2}{\rho g}$. 7. 315 ± 6%; 224 ± 4%.

Solutions to Exercises

1. For rise of water in the tube we have

$$2\pi r T \cos\theta = (\pi r^2 h \rho) g$$

(neglecting water in the meniscus)

or $$2T = r\rho g h \qquad (\because \theta = 0^\circ \text{ for water and glass})$$

or $$\frac{T}{r} = \tfrac{1}{2}\rho g h = \tfrac{1}{2} \times 1000 \times 9{\cdot}8 \times {\cdot}04 = 196.$$

When a hemispherical bubble is blown at the lower end,

pressure outside = p_0 (atmospheric pressure) + $\rho g l$

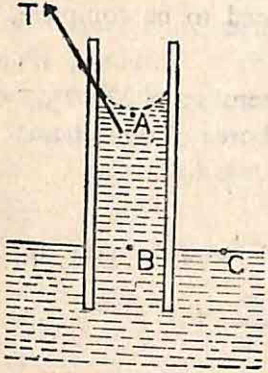
$\therefore$ pressure needed = p (pressure inside).

$$\text{Pressure difference} = p-(p_0+\rho gl)=\frac{2T}{r}.$$

or
$$p=p_0+\rho gl+\frac{2T}{r}$$
$$=p_0+1000\times 9{\cdot}8\times {\cdot}1+2\times 196$$
$$=p_0+980+392$$

$\therefore$ The gauge pressure needed = 1372 N/m². *Ans.*

2. The meniscus in between two plates is *cylindrical in shape.*



Pressure at A (lowest point or the meniscus

$$=p_0-\frac{T}{r}.$$

Pressure at B = pressure at $C = p_0$
= pressure at $A+\rho gh$

$$\therefore \quad p_0=p_0-\frac{T}{r}+\rho gh$$

or
$$h=\frac{T}{\rho gr}=\frac{2T}{\rho gd}.\ \textit{Ans.}$$

Alternative method : Force upward $=2lT\cos\theta=2lT$

$(\because \quad \theta=0°)$

Gravitational pull = (volume × density) $g=lhd\rho g$

$$\therefore \quad 2lT=lhd\rho g$$

or
$$h=\frac{2T}{\rho gd}.$$

3. Upward force $=(\pi r^2 l\rho)g$, where ρ = density of water.
Downward force $=mg+2\pi rT$.

$$\therefore \quad \pi r^2 l\rho g=mg+2\pi rT$$

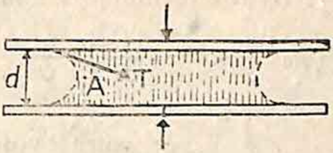
$$l=\frac{mg}{\pi r^2\rho g}+\frac{2\pi rT}{\pi r^2\rho g}=\frac{m}{\pi r^2\rho}+\frac{2T}{\rho gr}$$

$$\therefore \quad l=\frac{{\cdot}2\times 10^{-3}}{\pi\times {\cdot}14^2\times 10^{-4}\times 1000}+\frac{2\times {\cdot}07}{1000\times 9{\cdot}8\times {\cdot}14\times 10^{-2}}$$

$$=\frac{{\cdot}2\times 10^{-2}}{\pi\times {\cdot}14^2}+\frac{{\cdot}14}{9{\cdot}8\times {\cdot}14\times 10}={\cdot}032+{\cdot}010$$

$$={\cdot}042\text{ m}=4{\cdot}2\text{ cm}\ \textit{Ans.}$$

4. Let R be the radius of the circular layer of water. Then $\pi R^2 d \times \rho = m$.



Pressure at $A = p_0 - \frac{2T}{d}$

($\because$ meniscus is cylindrical in shape).

Thus pressure between the plates is less than the atmospheric pressure and so the plates are pressed together as if plates are attracted by themselves towards each other.

Force of attraction $= \triangle p \times$ area

$$= \frac{2T}{d} \times \pi R^2$$

$$= \frac{2T}{d} \times \frac{m}{d.\rho} = \frac{2Tm}{d^2.\rho}$$

$$= \frac{2 \times \cdot 2 \times 10^{-3} \times \cdot 07}{\cdot 01^2 \times 10^{-4} \times 1000}$$

$= 2{\cdot}8$ N. *Ans.*

5. Surface energy of 1000 droplets $= 1000(4\pi \times \cdot 05^2 \times 10^{-6}) \times \cdot 475$
$= 1{\cdot}49 \times 10^{-5}$ J.

Volume of 1000 droplets = volume of the drop

$1000 \times \frac{4\pi}{3}\left(\cdot 05\right)^3 = \frac{4\pi}{3} r^3$; or $r = 10 \times \cdot 5$ mm.

Surface energy of the drop $= 4\pi \times \cdot 5^2 \times 10^{-6} \times \cdot 475$
$= 1{\cdot}49 \times 10^{-6}$ J

$\therefore$ Loss in surface energy $= 1{\cdot}49 \times 10^{-5} - 1{\cdot}49 \times 10^{-6}$
$= 1{\cdot}34 \times 10^{-5}$ J. *Ans.*

6. For the air enclosed in the bubble we have from $pV = nRT$

$$\left(p_0 + \frac{8T}{d}\right) \cdot \frac{\pi}{6} d^3 = \left(\frac{p_0}{n} + \frac{8T}{rd}\right) \frac{\pi}{6} r^3 d^3$$

or
$$p_0 + \frac{8T}{d} = \frac{p_0 r^3}{n} + \frac{8Tr^2}{d}$$

or
$$p_0\left(1 - \frac{r^3}{n}\right) = \frac{8T}{d}(r^2 - 1)$$

or
$$T = \frac{1}{8} p_0 d \times \frac{1 - \frac{r^3}{n}}{r^2 - 1}. \quad \textit{Ans.}$$

7. By the formula $pV=mRT$ we have

$$\left(p_0+\rho_w gh-\frac{4T}{d}\right)\cdot\frac{\pi}{6}d^3=\left(p_0+\frac{4T}{nd}\right)\frac{\pi}{6}n^3d^3$$ (the bubble has only one surface)

pressure × volume = pressure × volume.

$$h=\frac{d(n^3-1)p_0+4T(n^2-1)}{\rho_w gd}$$

$$=\frac{4\times10^{-6}(1{\cdot}1^3-1)\times10^5+4\times{\cdot}07(1{\cdot}1^2-1)}{1000\times9{\cdot}8\times4\times10^{-6}}$$

$$=5\text{ m.}\quad \textit{Ans.}$$

8. See Ex. 6. $$h=\frac{2T\cos\theta}{\rho g}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$$

$$=\frac{2\times{\cdot}475\times\cos138^\circ}{13{\cdot}6\times1000\times9{\cdot}8}\left(\frac{2}{{\cdot}5\times10^{-3}}-\frac{2}{1\times10^{-3}}\right)$$

$$=\frac{-2\times{\cdot}475\times{\cdot}743\times2}{13{\cdot}6\times9{\cdot}8}(2-1)$$

$$=-10{\cdot}6\times10^{-3}\text{ m}=-10{\cdot}6\text{ mm.}\quad \textit{Ans.}$$

9. The pressure in the meniscus $=p_0-\dfrac{4T}{(d_2-d_1)}$

($\because$ the cylindrical meniscus has radius $\dfrac{d_2-d_1}{4}$.)

Pressure difference with flat surface $=p_0-\left(p_0-\dfrac{4T}{d_2-d_1}\right)$

$$=\frac{4T}{d_2-d_1}=\rho gh$$

$$\therefore\quad h=\frac{4T}{\rho g(d_2-d_1)}$$

$$=\frac{4\times{\cdot}07}{1000\times9{\cdot}8(2-1{\cdot}5)\times10^{-3}}=\frac{{\cdot}28}{9{\cdot}8\times{\cdot}5}={\cdot}057\text{ m}$$

$$=5{\cdot}7\text{ cm.}\quad \textit{Ans.}$$

Answers to Test Questions

1. (i) A liquid generally meets a solid surface at the angle of contact but not always. For example in an insufficiently long tube, the liquid meets at an angle greater than the angle of contact, if the liquid wets the solid.

(ii) $h+h=2h$.

(iii) The pressure in between plates is less than atmospheric pressure because of the fact that the pressure inside a bubble or drop is greater than the outside pressure. The atmospheric pressure from the sides presses the two plates together. No, the plates are pulled away from each other.

(iv) It is buoyed up by surface tension. In water it is buoyed down by surface tension.

(v) The time period of the pendulum is about 1·45. The percentage error of length $=\dfrac{1\,\text{mm}}{50\text{ cm}}\times 100=0{\cdot}2$. The time measurement must have % error of the same order. Let t be the total time to be considered. Then % error in time $=\dfrac{\cdot 1\text{ s}}{t\text{ s}}\times 100=0{\cdot}2$

or
$$t=50\text{ s}.$$

$\therefore$ No. of oscillations to be tried $=\dfrac{50}{1{\cdot}4}=35$. *Ans.*

(vi) cohesive; adhesive. (viii) (b). (ix) (c). (x) (a).

2. Pressure inside the bubble $=p_0+\rho gh+\dfrac{4T}{d}$

$\therefore$ The gauge pressure = pressure difference $=\rho gh+\dfrac{4T}{d}$

$$=13{\cdot}6\times 1000\times 9{\cdot}8\times 10^{-2}+\frac{4\times\cdot 5}{10^{-3}}=1333+2000=3333\text{ N/m}^2.$$

3. We have in general, $2\pi rT\cos\theta=(\pi r^2h'\rho)g$

or
$$2T\cos\theta=h'r\rho g$$

$\therefore$
$$2\times\cdot 07\cos\theta=(3\times 10^{-2})\times(1\times 10^{-3})\times 1000g$$

or
$$\cos\theta=\frac{9{\cdot}8}{14}.$$

Now, $r=R\cos\theta$ or $R=\dfrac{r}{\cos\theta}=\dfrac{10^{-3}}{\dfrac{9{\cdot}8}{14}}=1{\cdot}4\times 10^{-3}\text{ m}$

$$=1{\cdot}4\text{ mm. } \textit{Ans.}$$

4. Pressure inside the bubble $=p_0+\rho gh+\dfrac{4T}{d}$

$$=10^5+1000\times 9{\cdot}8\times 5+\frac{4\times 07}{4\times 10^{-6}}$$
$$=10^5+{\cdot}5\times 10^5+{\cdot}7\times 10^5$$
$$=2{\cdot}2\times 10^5\ \text{N/m}^2.\quad \textit{Ans.}$$

5. See Ex. 4. $F=\dfrac{2Tm}{\rho d^2}=\dfrac{2\times{\cdot}07\times 70\times 10^{-6}}{1000\times(1\times 10^{-4})^2}$

$$={\cdot}98\ \text{N}.\quad \textit{Ans.}$$

6. See Ex. 2. Heat energy produced $=\dfrac{2\pi T^2}{\rho g}$. *Ans.*

7. We have, $V=\dfrac{4\pi}{3}r^3.$

$$\therefore\quad \partial V=\frac{4\pi}{3}.\ 3r^2\,\partial r.$$

$$\therefore\quad \frac{\partial V}{V}=3\frac{\partial r}{r}$$

$\therefore$ % error in volume $=3\times$ % error in radius $=3\times 2=6.$

$$\text{Volume}=\frac{4\pi}{3}r^3=\frac{4\pi}{3}\times(4{\cdot}22)^3=315\ \text{cm}^3$$

$\therefore$ Volume $=(315\pm 6\%)$ cm³. *Ans.*

We have, $S=4\pi r^2.\partial S=8\pi r\partial r$

and $$\frac{\partial S}{S}=\frac{2\partial r}{r}$$

$\therefore$ % error in $s=2\times 2=4.$

Area $=4\pi r^2=4\pi\times 422^2=224$ cm²

$\therefore$ Area $=(224\pm 4\%)$ cm². *Ans.*

S. C. E. R. T. West
Library

BOOKS IN THIS SERIES

1. **Coaching at Your Home (PHYSICS) :** ***Part I*** **(Mechanics)**
2. **Coaching at Your Home (PHYSICS) :** ***Part II*** **(Heat, Sound & Light)**
3. **Coaching at Your Home (PHYSICS) :** ***Part III*** **(Electrostatics, Magnetostatics & Mod. Physics)**

BHARATI BHAWAN (P & D)
Patna * Ranchi * Muzaffarpur * Darbhanga

Coaching At Your Home : Physics : (Mechanics)
(7CC—45)

1110786

Rs. 32·75